# ETALE GROUPOIDS, ETA INVARIANTS AND INDEX THEORY

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Abstract. Let $\Gamma$ be a discrete finitely generated group. Let $\widehat{M} \to T$ be a $\Gamma$-equivariant fibration, with fibers diffeomorphic to a fixed even dimensional manifold with boundary $Z$. We assume that $\Gamma \to \widehat{M} \to \widehat{M}/\Gamma$ is a Galois covering of a compact manifold with boundary. Let $(D^+(\theta))_{\theta \in T}$ be a $\Gamma$-equivariant family of Dirac-type operators. Under the assumption that the boundary family is $L^2$-invertible , we define an index class in $K_0(C^0(T) \rtimes_r \Gamma)$. If, in addition, $\Gamma$ is of polynomial growth, we define higher indeces by pairing the index class with suitable cyclic cocycles. Our main result is then a formula for these higher indeces: the structure of the formula is as in the seminal work of Atiyah, Patodi and Singer, with an interior geometric contribution and a boundary contribution in the form of a higher eta invariant associated to the boundary family. Under similar assumptions we extend our theorem to any $G$-proper manifold, with $G$ an étale groupoid. We employ this generalization in order to establish a higher Atiyah-Patodi-Singer index formula on certain foliations with boundary. Fundamental to our work is a suitable generalization of Melrose $b$-pseudodifferential calculus as well as the superconnection proof of the index theorem on $G$-proper manifolds recently given by Gorokhovsky and Lott in [9].

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## 1. Introduction and main results

Connes' index theorem for $G$-proper manifolds [7], with $G$ an étale groupoid, unifies under a single statement most of the existing index theorems. For this introduction we shall focus on a particular case of such a theorem: $\Gamma$ is a discrete group acting on an even dimensional manifold $\widehat{M}$ and on a compact manifold $T$; $\widehat{M} \to T$ is a $\Gamma$-equivariant fibration and the action of $\Gamma$ on $\widehat{M}$ is free, properly discontinuous and cocompact, thus $\widehat{M}/\Gamma := M$ is a smooth compact manifold and $\Gamma \to \widehat{M} \to M$ is a $\Gamma$-Galois covering. (This is an example of $G$-proper manifold with $G$ equal to the groupoid $T \rtimes \Gamma$.)

If $T =$ point and $\Gamma = \{1\}$ we have a compact manifold and Connes' index theorem reduces to the Atiyah-Singer index theorem.

If $\Gamma = \{1\}$ we simply have a fibration and the theorem reduces to the Atiyah-Singer family index theorem.

Finally, if $T =$ point then we have a Galois covering and Connes' index theorem reduces to the Connes-Moscovici higher index theorem.

If $\dim T > 0$ and $\Gamma \neq \{1\}$, then Connes' index theorem can be seen as a higher foliation index theorem on the foliated manifold, $(M, \mathcal{F})$, obtained by foliating $M$ by the images of the fibers of $\widehat{M} \to T$. Recall that Connes' index theorem can be seen as a cohomological version of a K-theoretical statement, namely the Connes-Skandalis longitudinal index theorem for foliations [8].

In the case $\Gamma = \{1\}$, $T =$ point (compact closed manifolds), in the case $\Gamma = \{1\}$ (fibrations) and in the case $T =$ point (Galois coverings), the index theorems mentioned above

have been sharpened into *local index theorems.* For a single closed compact manifold this goes back to the work of McKean-Singer, Patodi, Gilkey and many others. In the case of a fibration the result is due to Bismut, see [3] [2], whereas in the case of Galois coverings the local version of the higher index theorem it is due to Lott, see [22]. Notice that for fibrations and Galois coverings the new proofs make use of the heat kernel associated to a superconnection. One of the most interesting outcome of these improved proofs is the possibility of extending the index theorem to manifolds with boundary. In the case of a single manifold and of a Dirac-type operator on it, such an index theorem is due to Atiyah-Patodi-Singer. For families of Dirac operators on manifolds with boundary, the theorem is due to Bismut and Cheeger and, more generally, to Melrose and Piazza ([4],[5],[27], [28]). Finally, for Galois coverings of a compact manifold with boundary, the result is due to the authors of the present article, see [17], [18], following a conjecture of Lott [23].
*Geometric applications* of these results have been given

- to the problem of defining higher signatures on manifolds with boundary [23] [24] and proving their homotopy invariance [15],
- to uniqueness problems in positive scalar curvature [20],
- to the problem of cut-and-paste invariance of Novikov higher signatures on closed manifolds [15] (see also [16] [10] [21]),
- to the homotopy invariance of the Atiyah-Patodi-Singer and Cheeger-Gromov rho-invariants for closed compact manifolds having a torsion-free fundamental group $\Gamma$ satisfying the bijectivity of the Baum-Connes map for $C^*_{\max}\Gamma$ [34], a result due originally to Keswani [13].

Recently Gorokhovsky and Lott have given a superconnection-heat-kernel proof of Connes' index theorem, see [9], and raised the question of extending such a theorem to manifolds with boundary. *The goal of this paper is to establish such a result, thus proving a higher Atiyah-Patodi-Singer higher index theorem on a $G$-proper manifold with boundary.* Geometric applications of our theorem will be considered in a future publication.

We shall now give a detailed description of the content of this paper. In Section 2 we introduce our basic geometric object: a $\Gamma$-equivariant fibration $\widehat{M} \to T$ with $\widehat{M}$ a manifold with boundary, $T$ a closed compact manifold and $\Gamma \to \widehat{M} \to \widehat{M}/\Gamma$ a $\Gamma$-Galois covering. This is done in Subsection 2.1 In Subsection 2.2 we introduce the groupoid $T \rtimes \Gamma$ and the associated $C^*$-algebra $C^*_r(T \rtimes \Gamma)$. The latter is simply the reduced cross-product algebra $C^0(T) \rtimes_r \Gamma$ and can be described as a suitable completion of the algebraic cross-product

$$C^\infty(T) \rtimes \Gamma = \{\sum_{\text{finite}} f_g(\theta) g\} \;\text{ with }\; g \cdot (f_h(\theta)h) := f_h(\theta \cdot g)gh\,.$$

We next explain how to associate to our $\Gamma$-equivariant fibration $\widehat{M} \to T$ some natural Sobolev $(C^0(T) \rtimes_r \Gamma)$-Hilbert modules $H^m_{b,C^0(T)\rtimes_r\Gamma}$; we also introduce $\Gamma$-equivariant families of Dirac-type operators and explain how such a $\Gamma$-equivariant family $\mathcal{D}$ defines a morphism of Sobolev $(C^0(T) \rtimes_r \Gamma)$-Hilbert modules $H^m_{b,C^0(T)\rtimes_r\Gamma} \to H^{m-1}_{b,C^0(T)\rtimes_r\Gamma}$; this is done in Subsections 2.3 and 2.4. A $\Gamma$-equivariant family of Dirac operators defines in a natural way a longitudinal elliptic operator on the foliated manifold $(M, \mathcal{F})$; for the sake of completeness we explain in Subsection 2.5 the connection between the noncommutative framework just explained and the one associated to such a longitudinal operator on $(M, \mathcal{F})$. We end Section 2.1

introducing our basic assumption on the family of operators, denoted $\mathcal{D}_0$, induced on the boundary by a $\Gamma$-equivariant family of Dirac operators $\mathcal{D}$; this is an invertibility assumption. We also draw some important consequences out of this assumption for the morphism of $(C^0(T) \rtimes_r \Gamma)$-Hilbert modules defined by $\mathcal{D}_0$.

In Section 3 we develop a $\Gamma$-equivariant fiber-$b$-calculus on $\widehat{M} \to T$ and we employ it in order to prove that, under our invertibility assumption, a $\Gamma$-equivariant family of Dirac operators defines an index class in $K_*(C^0(T) \rtimes_r \Gamma)$.

Once the index class is defined we need a way to extract numerical invariants out of it; these are the *higher indeces*. Our main concern is then to prove a Atiyah-Patodi-Singer-type formula for these higher indeces, equating a higher index with the sum of an interior geometric contribution and a boundary contribution.

In order to orient the reader, we recall the structure of these index theorems in the three special cases considered above. To fix the notation we assume that we are in the spin context and that our operators are Dirac operators acting on spinors.

If $T=$ point and $\Gamma = \{e\}$, then our $(T \rtimes \Gamma)$-proper manifold is simply equal to an even dimensional compact manifold with boundary $M$. In this case the index of the Atiyah-Patodi-Singer boundary value problem associated to the Dirac operator $D$ is expressed as follows:

$$\operatorname{ind}_{\mathrm{APS}}(D) = \int_M \widehat{A}(TM, \nabla^{\mathrm{LC}}) - \frac{\eta(D_0) + \dim \operatorname{Ker}(D_0)}{2}$$

with $\nabla^{\mathrm{LC}}$ equal to the Levi-Civita connection and $\eta(D_0)$ the eta invariant associated to the boundary operator $D_0$.

If $\Gamma = \{e\}$ then the $(T \rtimes \Gamma)$-proper manifold we are dealing with is a fibration

$$Z \to \widehat{M} \to T$$

with fibers diffeomeorphic to a compact spin manifold with boundary $Z$. If $\mathcal{D} = (D(\theta))_{\theta \in T}$ is a family of Dirac operators with invertible boundary family $\mathcal{D}_0 \equiv ((D(\theta)_0)_{\theta \in T}$ then there exists a well defined index class $\operatorname{Ind}_{\mathrm{APS}}(\mathcal{D}) \in K^0(T) = K_0(C^0(T))$. We extract higher indeces out of this index class by taking the Chern character $\operatorname{Ch}(\operatorname{Ind}_{\mathrm{APS}}(\mathcal{D})) \in H^*_{dR}(M)$ and coupling it with the homology class $[\Phi] \in H_*(M, \mathbb{R})$ defined by a closed current $\Phi$ . The index formula can already be given at the level of Chern character and reads:

$$\operatorname{Ch}(\operatorname{Ind}_{\mathrm{APS}}(\mathcal{D})) = \int_{\text{fiber}} \widehat{A}(T_V(\widehat{M}), \nabla^V) - \frac{1}{2}\widetilde{\eta}(\mathcal{D}_0) \quad \text{in} \quad H^*_{dR}(T)$$

with $\widetilde{\eta}(\mathcal{D}_0) \in \Omega^*(T) := C^\infty(T, \Lambda^* T)$ the Bismut-Cheeger eta form defined by the boundary family, $T_V(\widehat{M})$ the vertical tangent bundle to the fibration and $\nabla^V$ a certain connection on it. Notice that the right hand side of the index formula involves *smooth* differential forms.

If $T=$ point, then our $(T \rtimes \Gamma)$-proper manifold is a Galois covering $\Gamma \to \widehat{M} \to M$ of a compact manifold with boundary $M$ and the analytic object of interest to us is a $\Gamma$-invariant Dirac operator on $\widehat{M}$; we keep denoting this operator by $\mathcal{D}$. Under an invertibility assumption on the boundary operator $\mathcal{D}_0$ there is a well defined index class $\operatorname{Ind}_{\mathrm{APS}}(\mathcal{D}) \in K_0(C^*_r\Gamma)$. The reduced group $C^*$-algebra plays the role here of the continuous functions on the base of the fibration. As explained in [22] [23] this analogy can be pushed much further:

thus there exist a "smooth" subalgebra $\mathcal{B}^\infty_\Gamma$,

$$\mathbb{C}\Gamma \subset \mathcal{B}^\infty_\Gamma \subset C^*_r\Gamma\,,$$

playing the role of the $C^\infty$ function on the base of the fibration, and an algebra of "smooth" noncommutative differential forms $\widehat{\Omega}_*(\mathcal{B}^\infty_\Gamma)$. Under our invertibility assumption there is a well defined higher eta invariant $\widetilde{\eta}(\mathcal{D}_0)$ associated to the boundary family $\mathcal{D}_0$, an element in $\widehat{\Omega}_*(\mathcal{B}^\infty_\Gamma)$ modulo the closure of the graded commutator $[\widehat{\Omega}_*(\mathcal{B}^\infty_\Gamma),\widehat{\Omega}_*(\mathcal{B}^\infty_\Gamma)]$. The higher eta invariant was defined by Lott in [22] under the assumption that $\Gamma$ is of *polynomial growth*. For the general case see [24] and the Appendix in [19] . The "smooth" algebra $\mathcal{B}^\infty_\Gamma$ is in fact dense and holomorphically closed in $C^*_r\Gamma$, so that $K_*(\mathcal{B}^\infty_\Gamma) = K_*(C^*_r\Gamma)$. The graded differential algebra $\widehat{\Omega}_*(\mathcal{B}^\infty_\Gamma)$ defines noncommutative de Rham homology groups $\widehat{H}_*(\mathcal{B}^\infty_\Gamma)$ and there is a well defined Chern character $\mathrm{Ch} : K_*(\mathcal{B}^\infty_\Gamma) = K_*(C^*_r\Gamma) \to \widehat{H}_*(\mathcal{B}^\infty_\Gamma)$, see [12]. One defines higher indeces for $\mathcal{D}$ by pairing the Chern character of the index class, $\mathrm{Ch}(\mathrm{Ind}_{\mathrm{APS}}(\mathcal{D}))$, with an element in the cohomology of $\mathcal{B}^\infty_\Gamma$, i.e. with a *closed graded trace* $\Phi$ *on* $\widehat{\Omega}_*(\mathcal{B}^\infty_\Gamma)$. The result proved in [17] (see also [19] Appendix) gives a formula for these higher indeces; the formula already holds at the level of Chern character, as an equality in $\widehat{H}_*(\mathcal{B}^\infty_\Gamma)$, and reads

$$\mathrm{Ch}(\mathrm{Ind}_{\mathrm{APS}}(\mathcal{D})) = \left[\int_M \widehat{A}(M,\nabla^{\mathrm{LC}}) \wedge \omega - \frac{1}{2}\widetilde{\eta}(\mathcal{D}_0)\right] \quad \text{in} \quad \widehat{H}_*(\mathcal{B}^\infty_\Gamma)$$

with $\omega \in \Omega^*(M)\widehat{\otimes}\Omega_*(\mathbb{C}\Gamma)$ an explicit bi-form, see [22]. In particular, if $\Phi$ is a closed graded trace on $\Omega_*(\mathcal{B}^\infty_\Gamma)$ then the higher index $\mathrm{Ind}_\Phi(\mathcal{D}) :=< \mathrm{Ch}(\mathrm{Ind}_{\mathrm{APS}}(\mathcal{D})), [\Phi] >$ is well defined, $\omega_\Phi :=< \omega, \Phi >$ is a smooth differential form on $M$, $< \widetilde{\eta}(\mathcal{D}_0), \Phi >$ is also well defined and the following formula holds:

$$\mathrm{Ind}_\Phi(\mathcal{D}) = \int_M \widehat{A}(M,\nabla^{\mathrm{LC}}) \wedge \omega_\Phi - \frac{1}{2} < \widetilde{\eta}(\mathcal{D}_0), \Phi > . \tag{1}$$

We now back to the general case addressed in this paper: $\widehat{M} \to T$ is a $\Gamma$-equivariant fibration, $\Gamma \to \widehat{M} \to M$ is a Galois covering of a manifold with boundary and $\mathcal{D} = (D_\theta)_{\theta\in T}$ is a $\Gamma$-invariant family of Dirac operators. On the basis of the last two examples it is clear that in order to develop a higher Atiyah-Patodi-Singer index theorem we need first of all to fix a "smooth" subalgebra $\mathcal{T}^\infty$,

$$C^\infty(T) \rtimes \Gamma \subset \mathcal{T}^\infty \subset C^0(T) \rtimes_r \Gamma\,,$$

together with an algebra of smooth noncommutative differential forms. These two steps are in fact already necessary in the closed case, in order to develop a *local* higher index theory: we can thus follow closely the work of Gorokhovsky and Lott, although, for technical reasons having to do both with the convergence of the higher eta invariant and the construction of a suitable rapidly-decreasing-parametrix, we need to assume that $\Gamma$ is of polynomial growth. (For more on this, see the last remark in subsection 4.2.) The "smooth" subalgebra $\mathcal{T}^\infty$ is then defined in terms of infinite sums $\sum_{g\in\Gamma} f_g(\theta)g$ but with a growth condition on the coefficients $f_g(\theta) \in C^\infty(T)$:

$$\sup_{\theta\in T, g\in\Gamma} (|f_g|(1+\|g\|)^N) < \infty \ \ \forall N\,.$$

The noncomutative differential forms $\widehat{\Omega}(T, \mathcal{B}^\infty_\Gamma)$ are defined in terms of infinite sums

$$\sum \alpha_{g_0,g_1,\cdots,g_\ell} g_0 dg_1 \cdots dg_\ell\,, \quad \alpha_{g_0,g_1,\cdots,g_\ell} \in \Omega^k(T)$$

with a similar growth condition on the differential forms $\alpha_{g_0,g_1,\cdots,g_\ell} \in \Omega^k(T)$ and with the product taking into account the fact that $\Gamma$ acts on $T$. All this is explained in Section 5. The higher eta invariant associated to the boundary family $\mathcal{D}_0$, denoted $\widetilde{\eta}_{<e>}(\mathcal{D}_0)$, is an element in $\widehat{\Omega}_{*,<e>}(T, \mathcal{B}^\infty_\Gamma)$ (modulo graded commutators), the subalgebra of elements

$$\sum_{g_0 g_1 \cdots g_\ell = e} \alpha_{g_0,g_1,\cdots,g_\ell} g_0 dg_1 \cdots dg_\ell$$

concentrated on the trivial conjugacy class. The definition of $\widetilde{\eta}_{<e>}(\mathcal{D}_0)$ employs the Gorokhovsky-Lott superconnection, the heat kernel associated to it and a suitable supertrace; the latter is defined on the space of $\Gamma$-invariant families of smoothing operators with $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)$-coefficients and has values in $\Omega_{*,<e>}(T, \mathcal{B}^\infty_\Gamma)$ (modulo graded commutators). Let $\Phi$ a closed graded trace on $\widehat{\Omega}(T, \mathcal{B}^\infty_\Gamma)$; we assume that $\Phi$ is concentrated on the trivial conjugacy class, i.e. that it is non-zero only on the subalgebra $\widehat{\Omega}_{*,<e>}(T, \mathcal{B}^\infty_\Gamma)$. Then the number $< \widetilde{\eta}_{<e>}(\mathcal{D}_0), \Phi >$ is well-defined. One can also define a higher index $\mathrm{Ind}_\Phi(\mathcal{D})$. If $\Gamma$ acts by isometries on $T$, then $\mathcal{T}^\infty$ is stable under holomorphic functional calculus and the definition of $\mathrm{Ind}_\Phi(\mathcal{D})$ proceeds in a way similar to Galois coverings, i.e. through a noncommutative Chern character à la Karoubi. We explain all this in Section 7 and in Appendix C we prove the following higher Atiyah-Patodi-Singer index formula:

*there exists an explicit current $\omega_\Phi$ on $M := \widehat{M}/\Gamma$ such that*

$$\mathrm{Ind}_\Phi(\mathcal{D}) = < \widehat{A}(T\mathcal{F}, \nabla^{\mathrm{LC},\mathcal{F}}), \omega_\Phi > - \frac{1}{2} < \widetilde{\eta}_{<e>}(\mathcal{D}_0), \Phi > \;. \tag{2}$$

In the first summand on the right hand side the pairing between differential forms and currents appears.

If $\Gamma$ does not act by isometries then the definition of higher index $\mathrm{Ind}_\Phi(\mathcal{D})$ is more involved and the proof of formula 2 much more complicated. The definition is based on ideas of Connes ([7], page 229), Nistor [32] and Gorokhovsky-Lott; the proof is based on arguments used by Gorokhovsky-Lott in order to prove the local version of Connes' index theorem and on $b$-calculus techniques developed in Sections 16 and 6 of the present paper. The proof of formula 2 occupies all of Section 9.

The next section of the paper, Section 10, is devoted to a generalization of the above results to general $G$-proper manifolds, with $G$ an étale groupoid satisfying a suitable polynomial growth condition. We employ such a result in order to establish a higher Atiyah-Patodi-Singer index formula on certain foliations with boundary; this is done in Section 11.

Finally, in Appendix A we define the rapidly decreasing $b$-calculus, in Appendix B we define the space of $b$-smoothing operators with differential form coefficients and in Appendix C we give a proof of our theorem in the isometric case.

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## 2. $\Gamma$-equivariant fibrations

### 2.1. Geometric data.

Let $\Gamma$ be a finitely generated discrete group. Let $T$ be a smooth closed compact connected manifold on which $\Gamma$ acts on the right. Let $\widehat{M}$ be a manifold with boundary on which $\Gamma$ acts freely, properly and cocompactly on the right: the quotient space $M = \widehat{M}/\Gamma$ is thus a smooth compact manifold with boundary. We assume that $\widehat{M}$ fibers over $T$ and that the resulting fibration

$$\pi : \widehat{M} \to T$$

is a $\Gamma$-*equivariant fibration* with fibers $\pi^{-1}(\theta), \theta \in T$, that are transverse to $\partial\widehat{M}$ and of dimension $2k$. Notice that each fiber is a smooth manifold with boundary; we shall also denote the typical fiber of $\pi : \widehat{M} \to T$ by $Z$. We choose a $\Gamma$-invariant exact $b$-metric [26] on the vertical $b$-tangent bundle ${}^bTZ$. Finally, we assume the existence of a $\Gamma-$equivariant spin structure on ${}^bTZ$ that is fixed once and for all. We denote by $S^Z \to \widehat{M}$ the associated spinor bundle.

The compact manifold with boundary $M$ inherits a foliation $\mathcal{F}$, with leaves equal to the image of the fibres of $\pi : \widehat{M} \to T$ under the quotient map $\widehat{M} \to M = \widehat{M}/\Gamma$. Notice that the foliation $\mathcal{F}$ is transverse to the boundary of $M$.

**Example.** Let $X$ be a compact manifold with boundary and let $\Gamma \to \widetilde{X} \to X$ be a Galois cover of $X$. Let $T$ be a smooth compact manifold on which $\Gamma$ acts by diffeomorphisms. We consider $\widehat{M} = \widetilde{X} \times T$, $\pi =$ projection onto the second factor, $M = \widetilde{X} \times_\Gamma T := (\widetilde{X} \times T)/\Gamma$ where we let $\Gamma$ act on $\widetilde{X} \times T$ diagonally. The leaves of the foliation $\mathcal{F}$ are the images of the manifolds $\widetilde{X} \times \{\theta\}$, $\theta \in T$. The foliated manifold $(M, \mathcal{F})$ is usually refered to as a *foliated $T$-bundle.*

As a particular example of this construction consider a closed smooth closed riemann surface $\Sigma$ of genus $g > 1$ and let $\Gamma = \pi_1(\Sigma)$, a discrete subgroup of $PSL(2, \mathbb{R})$. Let $\{p_1, \dots, p_k\}$ points in $\Sigma$ and let $D_j \subset \Sigma$ be a small open disc around $p_j$. Let $D = \cup_{j=1}^k D_j$. Then we can consider $X := \Sigma \setminus D$, $\Gamma \to \widetilde{X} \to X$ the Galois cover induced by the universal cover $\mathbb{H}^2 \to \Sigma$, $T = S^1$, with $\Gamma$ acting on $S^1$ by fractional linear transformations.

### 2.2. The groupoid $T \rtimes \Gamma$.

We consider the groupoid $G = T \rtimes \Gamma$ with set of morphisms $G^{(1)} = T \times \Gamma$ and base $G^{(0)} = T$. The range and source maps are respectively given by:

$$\forall (\theta, g) \in T \times \Gamma,\ r(\theta, g) = \theta, \quad s(\theta, g) = \theta \cdot g\,.$$

The composition is defined as follows:

$$(\theta, g) \cdot (\theta', g') = (\theta, gg') \text{ if } \theta' = \theta g\,.$$

The inverse of $(\theta, g)$ is $(\theta g, g^{-1})$. For more about groupoids we refer the reader to [7].

The algebraic cross-product $C_c^\infty(T) \rtimes \Gamma$ is, by definition, the set of functions $\sum_{g\in\Gamma} t_g(\theta)g$ such that only a finite number of the $t_g(\cdot) \in C_c^\infty(T)$ do not vanish identically. We shall identify any function $f$ having compact support

$$f : T \rtimes \Gamma \to \mathbb{C}$$

$$(\theta, g) \to f(\theta, g)$$

with $\sum_{g\in\Gamma} f(\theta, g)g$. Then one has:

$$\sum_{g'\in\Gamma} f'(\theta, g')g' \cdot \sum_{g\in\Gamma} f(\theta, g)g \;=\; \sum_{h\in\Gamma} \left( \sum_{g\in\Gamma} f'(\theta, g')\, f(\theta \cdot g', (g')^{-1}h) \right) h$$

where where we recall that

$$g' \cdot (f(\theta, g)g) = f(\theta \cdot g', g)g'g\,.$$

The algebraic cross-product $C_c^\infty(T) \rtimes \Gamma$ will also be denoted by $C_c^\infty(T \rtimes \Gamma)$. One can introduce the reduced $C^*$-algebra $C_r^*(T \rtimes \Gamma)$ associated to the groupoid $T \rtimes \Gamma$ as a suitable completion of the algebraic cross product $C_c^\infty(T) \rtimes \Gamma$. See [7]. It is well known, and easy to check, that there is a natural isomorphism between the reduced $C^*-$algebra $C_r^*(T \rtimes \Gamma)$ of the groupoid $T \rtimes \Gamma$ and the cross-product algebra $C^0(T) \rtimes_r \Gamma$ (see, for example, [30]) ; we shall henceforth identify these two $C^*$-algebras.

What we have described is an example of a proper cocompact $G$-manifold $P$ with $G$ an étale groupoid, see [7] (page 137) for the definition. In our case

$$G = T \rtimes \Gamma\,, \qquad G^{(0)} = T\,, \qquad (\alpha : P \to G^{(0)}) \equiv (\pi : \widehat{M} \to T).$$

We shall deal with the general case in Section 10.

### 2.3. $\Gamma$-equivariant families of operators.

We consider a $\Gamma$-equivariant complex hermitian vector bundle $\widehat{V} \to \widehat{M}$ endowed with a $\Gamma-$invariant $b-$hermitian connection $\widehat{\nabla}$ satisfying $\widehat{\nabla}_{x\partial_x} = 0$ on the boundary $\partial\widehat{M}$. We then set $\widehat{E} = S^Z \otimes \widehat{V} = \widehat{E}^+ \oplus \widehat{E}^-$ which defines a smooth $\Gamma-$invariant family of $\mathbb{Z}_2-$graded hermitian Clifford modules on the fibers $\pi^{-1}(\theta)$, $\theta \in T$. We then get a smooth family of $\Gamma-$invariant $\mathbb{Z}_2-$graded Dirac type operators

$$D(\theta) = \begin{pmatrix} 0 & D^-(\theta) \\ D^+(\theta) & 0 \end{pmatrix},\ \theta \in T$$

acting fiberwise on $C_c^\infty(\widehat{M}, \widehat{E})$. Moreover in a collar neighborhood ($\sim [0,1] \times \partial\pi^{-1}(\theta) = \{(x, y)\}$) of $\partial\pi^{-1}(\theta)$ we may write:

$$D^+(\theta) = \sigma(x\partial_x + D_0(\theta)\,)$$

where $D_0(\theta)$ is the induced boundary Dirac type operator acting on

$$C^\infty(\partial\pi^{-1}(\theta),\, \widehat{E}^+_{|_{\partial\pi^{-1}(\theta)}}\,).$$

Observe that our family can also be thought as a longitudinal operator on $(M, \mathcal{F})$ acting on the sections of $E := \widehat{E}/\Gamma$.

2.4. $C^*_r(T \rtimes \Gamma)$**-Hilbert modules.**

We shall now describe how the $\Gamma$-equivariant family $(D(\theta))_{\theta\in T}$ defines a $C(T)\rtimes_r \Gamma-$linear operator on suitable $C(T) \rtimes_r \Gamma-$Hilbert modules.
Recall that $\widehat{E}$ is a $\Gamma-$equivariant hermitian vector bundle over $\widehat{M}$ so that for each $(g, p) \in \Gamma \times \widehat{M}$ there is a unitary linear map $U_{g,p} : \widehat{E}_{p\cdot g} \to \widehat{E}_p$. Then we endow $C^\infty_c(\widehat{M}, \widehat{E})$ with the structure of a left $C^\infty_c(T) \rtimes \Gamma-$module by setting for any $s \in C^\infty_c(\widehat{M}, \widehat{E})$ and $\sum_{g\in\Gamma} f(., g)g \in C^\infty_c(T) \rtimes \Gamma$

$$\forall p \in \widehat{M}, \quad \left(\sum_{g\in\Gamma} f(., g)g \cdot s\right)(p) := \sum_{g\in\Gamma} f(\pi(p), g)(R^*_g s)(p) \tag{3}$$

where $(R^*_g s)(p) = U_{p,g}(s(p \cdot g))$; observe that $R^*_g \circ R^*_{g'} = R^*_{gg'}$.

**Lemma 1.** *The family of Dirac operators $(D(\theta))_{\theta\in T}$ acting fiberwise on $C^\infty_c(\widehat{M}, \widehat{E})$ defines a left $C^\infty_c(T \rtimes \Gamma)-$linear endomorphism $\mathcal{D}$ of $C^\infty_c(\widehat{M}, \widehat{E})$.*

*Proof.* Using the above notations we have:

$$\mathcal{D}\big(\sum_{g\in\Gamma} f(., g)g \cdot s\big)(p) = \sum_{g\in\Gamma} f(\pi(p), g)\, D(\pi(p))(R^*_g s)\,(p)$$

where we have used the fact that $(D(\theta))_{\theta\in T}$ is a family of operators, i.e. commutes with the natural action of $C^\infty(T)$. Since the family $(D(\theta))_{\theta\in T}$ is $\Gamma-$equivariant, the right hand is by definition equal to

$$\left(\sum_{g\in\Gamma} f(., g)g \cdot \mathcal{D}(s)\right)(p)$$

which proves the lemma. □

Let $\dot{C}^\infty_c(\widehat{M}, \widehat{E})$ the space of sections of compact support vanishing of infinite order at $\partial\widehat{M}$. We define the $C^0(T) \rtimes_r \Gamma-$hermitian product of two sections $s$ and $s'$ of $\dot{C}^\infty_c(\widehat{M}, \widehat{E})$ by setting:

$$\langle s; s'\rangle \;=\; \sum_{g\in\Gamma} \langle s; s'\rangle(\theta, g)g \tag{4}$$

where $\forall(\theta, g) \in T \times \Gamma$:

$$\langle s; s'\rangle(\theta, g) \;=\; \int_{\pi^{-1}(\theta.g)} < R^*_{g^{-1}}(s)(y); s'(y) >_{\widehat{E}} d\mathrm{Vol}^b_{\pi^{-1}(\theta\cdot g)}(y)$$

where $d\mathrm{Vol}^b_{\pi^{-1}(\theta\cdot g)}(y)$ denotes the b$-$riemannian density in the fiber.

We shall sometimes write, more shortly, $d\mathrm{Vol}^b(y)$ (the domain of integration appears already in the integral). With the definition (4) one has:

**Lemma 2.**

$$\langle t \cdot s; s'\rangle \;=\; t \cdot \langle s; s'\rangle, \qquad \forall t \in C^\infty_c(T \rtimes \Gamma)\,.$$

*Proof.* We may assume that $t = t_\gamma(.)\gamma$ for $\gamma \in \Gamma$. Then for any $(\theta, g) \in T \times \Gamma$ we have:

$$\langle t_\gamma(.)\gamma \cdot s; s'\rangle(\theta, g) = \int_{\pi^{-1}(\theta\cdot g)} < R^*_{g^{-1}}(t_\gamma(\pi(.))R^*_\gamma s)(y); s'(y) >_{\widehat{E}} d\mathrm{Vol}^b_{\pi^{-1}(\theta\cdot g)}(y)$$

$$= t_\gamma(\pi(y \cdot g^{-1})) \int_{\pi^{-1}(\theta\cdot g)} < (R^*_{g^{-1}\gamma}s)(y); s'(y) >_{\widehat{E}} d\mathrm{Vol}^b_{\pi^{-1}(\theta g)}(y)$$

but this last term is equal to $\langle s; s'\rangle(\theta \cdot \gamma, \gamma^{-1}g)t_\gamma(\theta)$ so that

$$\sum_{g\in\Gamma} \langle t_\gamma(.)\gamma \cdot s; s'\rangle(\theta, g)\, g = \sum_{g\in\Gamma} \langle s; s'\rangle(\theta.\gamma, \gamma^{-1}g)t_\gamma(\theta)\gamma\, (\gamma^{-1}g).$$

But we can write the right hand side as

$$\sum_{h\in\Gamma} < s; s' > (\theta\gamma, h)t_\gamma(\pi(\theta))\gamma h$$

which is equal to

$$t_\gamma(\pi(\theta))\gamma \sum_{h\in\Gamma} < s; s' > (\theta, h)h$$

which is exactly $t \cdot \langle s; s'\rangle$ as required. The lemma is proved. □

**Definition 1.** *The completion of $\dot{C}^\infty_c(\widehat{M}, \widehat{E})$ with respect to the hermitian scalar product (4) defines a $C^0(T) \rtimes_r \Gamma-$Hilbert module denoted by $L^2_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$.*

Similarly one defines the b-Sobolev spaces $H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ for $m \in \mathbb{Z}$ where the $L^2$-condition is defined with respect to the space $\mathrm{Diff}^m_{b,\rtimes}(\widehat{M}, \widehat{E})$ of $\Gamma-$invariant fiberwise $b-$differential operators of order $m$. If $m \geq 0$ then

$$H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}) = \{s \in L^2_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}); Pu \in L^2_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})\, \forall P \in \mathrm{Diff}^m_{b,\rtimes}(\widehat{M}, \widehat{E})\}$$

If $m < 0$ then

$$H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}) = \mathrm{Diff}^{-m}_{b,\rtimes}(\widehat{M}, \widehat{E})\, L^2_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$$

Our $\Gamma$-equivariant family $(D(\theta))_{\theta\in T}$ defines a morphism

$$\mathcal{D} : \dot{C}^\infty_c(\widehat{M}, \widehat{E}) \longrightarrow \dot{C}^\infty_c(\widehat{M}, \widehat{E})$$

of pre-hilbert $(C^0(T) \rtimes_r \Gamma)$-modules which extends for each $m \in \mathbb{Z}$ to a bounded morphism

$$H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}) \longrightarrow H^{m-1}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})\,.$$

We end this section by a useful

**Lemma 3.** *For each $m \in \mathbb{Z}$, the Hilbert-modules $H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ are standard i.e. isomorphic to $l^2(C^0(T) \rtimes_r \Gamma)$.*

*Proof.* We shall only consider the case $m = 0$. There exists a positive integer $N$ and open connected subsets $U_i \subset U'_i$ of $T$ $(1 \leq i \leq N)$ with the following properties. Each $U_i$ is relatively compact in $U_i$, $\cup_{1\leq i\leq N} U_i = T$, and for each $i \in \{1, \ldots, N\}$ the restriction of the fibration $\pi$ to $\pi^{-1}(U'_i)$ is trivial: $\pi^{-1}(U'_i) \simeq U'_i \times Z$. Denote by $\mu$ the given $\Gamma-$invariant riemannian measure and $v_f$ the volume of a fundamental domain for the action of $\Gamma$ on $\widehat{M}$.

Then using an induction argument on $i \in \{1, \ldots, N\}$, we may find an open connected subset $W \subset Z$ and open subsets $V'_i \subset \pi^{-1}(U'_i)$ $(1 \leq i \leq N)$ with the following three properties:
1) $V'_i \simeq U'_i \times W$ for each $i \in \{1, \ldots, N\}$ and $\mu(\cup_{1 \leq i \leq N} V'_i) \leq \frac{1}{2} v_f$
2) $\forall \gamma \in \Gamma \setminus \{e\}, \forall i \in \{1, \ldots, N\}, \; V'_i \cdot \gamma \cap V'_i = \emptyset$.
3) $\forall \gamma \in \Gamma, \forall i, j \in \{1, \ldots, N\}$, with $i \neq j, \; V'_i \cdot \gamma \cap V'_j = \emptyset$.

Then, for each $i \in \{1, \ldots, N\}$, we may find a sequence $(e^i_l)_{l \in \mathbb{N}}$ of elements of $C^0(\overline{V'_i}, \widehat{E})$, vanishing in a neighborhood of the boundary, such that for any $\theta \in U'_i$ and any $l, l' \in \mathbb{N}$:

$$\int_{\pi^{-1}(\theta)} < e^i_l(y); e^i_{l'}(y) >_{\widehat{E}} d\mathrm{Vol}^b_{\pi^{-1}(\theta)}(y) = \delta_{l,l'}.$$

Next we consider for each $i \in \{1, \ldots, N\}$ a smooth function $\phi_i \in C^\infty_c(U'_i)$ such that $\phi_i = 1$ on $U_i$ and we set for each $l \in \mathbb{N}$:

$$\xi_l \; = \; \frac{1}{\sqrt{\sum_{i=1}^N \phi_i^2}} \sum_{i=1}^n \phi_i e^i_l \in L^2_{b, C^0(T) \rtimes_r \Gamma}(V'_i, \widehat{E})$$

Then, using the above properties, one checks that for any $l, l' \in \mathbb{N}$ one has

$$< \xi_l; \xi_{l'} >= \delta_{l,l'} \text{ in } C^0(T) \rtimes_r \Gamma.$$

Then the $\xi_l$, $l \in \mathbb{N}$ generate a closed Hilbert submodule $\mathcal{H}$ of $L^2_{b, C^0(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E})$ which is isomorphic to $l^2(C^0(T) \rtimes_r \Gamma)$ and such that

$$L^2_{b, C^0(T) \rtimes_r \Gamma}(\widehat{M}, \widehat{E}) = \mathcal{H} \oplus \mathcal{H}^\perp.$$

The Lemma therefore follows from Kasparov's stabilization theorem, see [1]. □

2.5. **Foliations.**

Now, for the convenience of the reader, we connect the noncommutative picture we have described above with the more familiar noncommutative picture arising from the foliated manifold $(M, \mathcal{F})$, as in the work of A. Connes. We follow closely Morioshi-Natsume [31]. In this subsection only we assume, for simplicity, that $\widehat{M} = \widetilde{M} \times T$, the fibration $\pi$ is given by the projection $\pi : \widetilde{M} \times T \to T$ and the group $\Gamma$ acts on the manifold with boundary $\widetilde{M}$ (and on $T$) in such a way that the quotient $\widetilde{M}/\Gamma$ is a smooth compact manifold with boundary. So the product foliation $\widetilde{M} \times \{\theta\}$, $\theta \in T$ descends to a foliation $\mathcal{F}$ on $M$. We shall assume that the $\Gamma-$action on $T$ satisfies the following:

**Condition.** For $\gamma \in \Gamma$, if there exists an open subset $U$ of $T$ such that $\gamma(\theta) = \theta$ for any $\theta \in U$, then $\gamma$ is the identity element of $\Gamma$.

This condition guarantees that the holonomy groupoid $G$ of $(M, \mathcal{F})$ is Hausdorff and given by:

$$G \simeq (\widetilde{M} \times \widetilde{M} \times T)/\Gamma$$

where $\gamma \in \Gamma$ acts by $(y, z, \theta).\gamma = (y.\gamma, z.\gamma, \theta.\gamma)$. The source map $s$ and the range map $r$ are given by:

$$r([y, z, \theta]) = [y, \theta], \quad s([y, z, \theta]) = [z, \theta].$$

Fix a $b-$metric on $\widetilde{M}/\Gamma$, the lifting to $\widetilde{M}$ of this $b-$metric induces a left Haar system $\{\nu^y\}$ on the groupoid $G$, see [36]. We now recall the definition of the $C^*-$algebra of the

foliation $\mathcal{F}$ with coefficient in the hermitian vector bundle $E \to M$ whose lift to $\widehat{M}$ is $\widehat{E}$. Denote by $\dot{C}_c^\infty(G,E)$ the space of all compactly supported smooth sections of the bundle $(s^*(E))^* \otimes r^*(E)$ vanishing of infinite order at the boundary. The space $\dot{C}_c^\infty(G,E)$ has a $*-$algebra structure:

$$(f_1 * f_2)(\gamma) = \int_{G^{r(\gamma)}} f_1(\gamma')f_2({\gamma'}^{-1}\gamma)\, d\nu^{r(\gamma)}(\gamma'),$$
$$f^*(\gamma) = (f(\gamma^{-1}))^*.$$

where we recall that $G^x := \{\gamma \in G; r(\gamma) = x\}$. Now let $\lambda_\theta, \theta \in T$ be the strictly positive lifted $b-$density on $\widetilde{M} \times \{\theta\}$ and set

$$H_\theta = L^2_b(\widetilde{M} \times \{\theta\}; \widehat{E}_{|\widetilde{M}\times\{\theta\}},\, \lambda_\theta).$$

Then the collection $\mathcal{H} = (H_\theta)_{\theta\in T}$ together with the space of compactly supported vanishing near the boundary continuous sections of the bundle $\widehat{E}$ over $\widetilde{M} \times T$, defines a continuous field of Hilbert spaces over $T$. The $\Gamma-$action on $\widetilde{M} \times T$ and $\widehat{E}$ gives rise to an action on $\mathcal{H}$ denoted by $\xi \to \gamma\xi$ for $\gamma \in \Gamma$ and $\xi$ a section of $\mathcal{H}$. The space $\mathrm{End}_\Gamma(H)$ of $\Gamma-$equivariant bounded measurable fields of operators $K = (K_\theta), K_\theta \in B(H_\theta)$, is a $C^*-$algebra, where the norm is given by

$$||K|| = \sup\{||K_\theta||;\, \theta \in T\}.$$

There is a faithful representation $\rho : \dot{C}_c^\infty(G,E) \to \mathrm{End}_\Gamma(\mathcal{H})$ defined by

$$\forall f \in \dot{C}_c^\infty(G,E), \quad [\rho(f)_\theta\xi](y) = \int f(y,z,\theta)\xi(z)d\lambda_\theta(z),$$

where $\xi \in H_\theta$. The norm closure of $\dot{C}_c^\infty(G,E)$ with respect to the norm

$$||f|| = ||\rho(f)|| = \sup\{\, ||\rho(f)_\theta||;\ \theta \in T\}$$

is by definition the $C^*-$algebra $C^*(M,\mathcal{F},E)$ of the foliation $(M,\mathcal{F})$ with coefficient $E$.

Now we define a right action of $C_c^\infty(T \rtimes \Gamma)$ on the space $\dot{C}_c^\infty(\widetilde{M} \times T; \widehat{E})$ of sections vanishing of infinite order at the boundary as follows.

$$(\xi f)(z,\theta) = \sum_{\gamma\in\Gamma} f(\theta\gamma^{-1},\gamma)\xi(z\gamma^{-1},\theta\gamma^{-1}), \quad \xi \in \dot{C}_c^\infty(\widetilde{M} \times T; \widehat{E}), \quad f \in C_c^\infty(T \rtimes \Gamma).$$

A $C_c^\infty(T \rtimes \Gamma)-$valued inner product $< .; . >$ on $\dot{C}_c^\infty(\widetilde{M} \times T; \widehat{E})$ is defined by

$$< \xi_1; \xi_2 > (\theta,\gamma) = \int_{\widetilde{M}\times\{\theta\}} < \xi_1(z,\theta); \xi_2(z\gamma,\theta\gamma) >_{\widehat{E}} d\lambda_\theta(z),$$

where $< .; . >_{\widehat{E}}$ is the hermitian scalar product of the vector bundle $\widehat{E}$. Recall now that $C_c^\infty(T \rtimes \Gamma)$ is a dense sub-algebra of the reduced $C^*-$algebra $C(T) \rtimes_r \Gamma$. Denote as before by $L^2_{b,C^0(T)\rtimes_r\Gamma}(\widetilde{M} \times T, \widehat{E})$ the $C(T) \rtimes_r \Gamma-$Hilbert module obtained by taking the $C(T) \rtimes_r \Gamma-$completion of $\dot{C}_c^\infty(\widetilde{M} \times T; \widehat{E})$ with respect to the norm $||\xi|| = \sqrt{|| < \xi; \xi > ||_{C(T)\rtimes_r\Gamma}}$.

The fundamental point here is then the following: the left action of $\dot{C}_c^\infty(G; \widehat{E})$ on $L^2_{b,C^0(T)\rtimes_r\Gamma}(\widetilde{M}\times T, \widehat{E})$ given by

$$(f * \xi)(y,\theta) = \int_{\widetilde{M}\times\theta} f(y,z,\theta)\xi(z,\theta)d\lambda_\theta(z)$$

(where we have used the identification of $G$ with $(\widetilde{M} \times \widetilde{M} \times T)/\Gamma)$, extends to a faithful representation of $C^*(M, \mathcal{F}, E)$ and the image of this representation is precisely the space $\mathbb{K}_{C(T)\rtimes_r\Gamma}(L^2_{b,C^0(T)\rtimes_r\Gamma}(\widetilde{M}\times T, \widehat{E}))$ of $(C(T)\rtimes_r\Gamma)-$compact operators on $L^2_{b,C^0(T)\rtimes_r\Gamma}(\widetilde{M}\times T, \widehat{E})$. The proof of this fact is exactly as in [31], Proposition 2.4.

2.6. **Invertible families.**

We consider a closed $\Gamma$-manifold $\widehat{N}$ fibering over $T$. The fibration $\pi : \widehat{N} \longrightarrow T$ is assumed to be $\Gamma$-equivariant and the fibers to be of dimension $2k-1$. One example of this situation is given by the boundary $\partial\widehat{M} \to T$ of a fibration with boundary as in the previous subsections, but we want to proceed in full generality now. We endow $\widehat{N}$ with a $\Gamma$-invariant riemannian metric and we denote by $d( \, , \, )$ the induced distance. The group $\Gamma$ is assumed to act freely properly and cocompactly on $\widehat{N}$ so that the quotient $N := \widehat{N}/\Gamma$ is a compact smooth manifold. Moreover we assume that the fibers of $\pi$ carry a $\Gamma-$equivariant spin structure and denote by $S \to \widehat{N}$ the associated spinor bundle. We also consider a $\Gamma$-equivariant hermitian complex vector bundle $\widehat{V} \to \widehat{N}$ endowed with a $\Gamma-$invariant connection. We then set $\widehat{F} = S \otimes \widehat{V}$ on $\widehat{N}$. We get in this way a $\Gamma$-equivariant family $(D_0(\theta))_{\theta\in T}$ acting along the fibers of $\pi : \widehat{N} \longrightarrow T$.

Some of the results in this paper are proved under the following assumption:

**Hypothesis A.** *There exists a real $\epsilon > 0$ such that for any $\theta \in T$, the $L^2$-spectrum of $D_0(\theta)$ acting on $L^2(\pi^{-1}(\theta), \widehat{F}_{|_{\pi^{-1}(\theta)}})$ does not meet $]-\epsilon, \epsilon[$.*

**Example.**
Let $Y$ be a *spin* compact manifold without boundary and let $\Gamma \to \widetilde{Y} \to Y$ be a Galois cover of $Y$. Let $T$ be a smooth compact manifold on which $\Gamma$ acts by diffeomorphisms. We consider $\widehat{N} = \widetilde{Y} \times T$, $\pi =$ projection onto the second factor, $N = \widetilde{Y} \times_\Gamma T := (\widetilde{Y} \times T)/\Gamma$ where we let $\Gamma$ act on $\widetilde{Y} \times T$ diagonally. We consider a hermitian vector bundle $V$ on $N$ with hermitian connection $\nabla^V$ and its lift $\widehat{V}$ on $\widehat{N}$. We denote by $\widehat{V}_\theta$ the restriction of $\widehat{V}$ to $\widetilde{Y} \times \{\theta\}$. We endow $\widehat{V}$ with the pulled-back metric and with the pulled-back connection. We now assume that $Y$ admits a metric of *positive scalar curvature* and we fix such a metric $g_Y$; we let $g_{\widetilde{Y}}$ be the pulled-back metric. Let $\widetilde{D}$ be the associated Dirac operator on $\widetilde{Y}$. We consider the family

$$D(\theta) = \widetilde{D}_{\widehat{V}_\theta}$$

where on the right hand side we are considering the twisted Dirac operator. This is a $\Gamma$-equivariant family; if the scalar curvature of $g_Y$ is big enough, then, by Lichnerowicz formula, we easily obtain that $(D(\theta))_{\theta\in T}$ satisfies Hypothesis A.
We now give a particular example showing the existence of a *boundary* family satisfying Hypothesis A.
Let $\Gamma$ be a finitely presented group. There exists a $(2k+1)$-dimensional closed spin manifold $N$ such that $2k+1 \geq 7$ and $\Gamma = \pi_1(N)$. We consider $X_1 = B^9 \times N$, with $B^9$ the unit ball in the standard euclidean space $\mathbb{R}^9$. We perform surgery along a submanifold of $X_1 \setminus \partial X_1$ of codimension at least 3 and we obtain a manifold with boundary equal to $Y = S^8 \times N$ and fundamental group $\Gamma$. Let $\widetilde{X}$ be the universal cover of $X$, let $\widetilde{Y} = \partial\widetilde{X}$ and let $\widehat{M} = \widetilde{X} \times T$, $M = (\widetilde{X} \times T)/\Gamma$. Fix a hermitian bundle $V$ on $M$ and consider its pull-back $\widehat{V}$ to $\widehat{M}$, as in

the example given above. Consider now the family

$$D(\theta) = \widetilde{D}_{\widehat{V}_\theta}$$

with $\widetilde{D}$ denoting now the Dirac operator on $\widetilde{X}$. This is a family acting on the section of the bundle $\widehat{E} = S \otimes \widehat{V}$ with $S$ equal to the spinor bundle on $\widetilde{X}$. Let $\widetilde{D}_0$ be the Dirac operator on $\widetilde{Y} = \partial \widetilde{X}$. The boundary family associated to $(D^+(\theta))_{\theta \in T}$ is given by the operator $\widetilde{D}_0$ twisted by $(\widehat{V}_\theta)|_{\partial \widetilde{X} \times \{\theta\}}$. Let us fix a metric of positive scalar curvature on $Y$ and let us extend this metric to $X$; if the scalar curvature of $Y$ is big enough then the boundary family $(D_0(\theta))_{\theta \in T}$ will satisfy Hypothesis A. A particular example of this situation is $\Gamma = \mathbb{Z}^k$ acting on $T = (S^1)^k$, the $k$-dimensional torus, by

$$(n_1, \dots, n_k) \cdot (e^{i\theta_1}, \dots, e^{i\theta_k}) = (e^{i\theta_1 + 2i\pi n_1 \alpha_1}, \dots, e^{i\theta_k + 2i\pi n_k \alpha_k})$$

where $\alpha_1, \dots, \alpha_k$ are $k$ fixed irrational numbers.
Another particular example is given by taking $\Gamma = \pi_1(\Sigma_g)$, where $\Sigma_g$ equal to a closed riemann surface of genus $g \geq 2$. Then one can take $T = S^1$ with $\Gamma$ acting by fractional linear tranformations.

We introduce the space $\Psi^*_\pi(\widehat{N}, \widehat{F})$ of smoothly varying families of pseudodifferential operators along the fibers. The Schwartz kernel of an element $P \in \Psi^*_\pi(\widehat{N}, \widehat{F})$ is a distribution on the fibred product $\widehat{N} \times_\pi \widehat{N}$ with the usual conormal singularities on the fibre diagonal. The space of $\Gamma$-equivariant elements in $\Psi^*_\pi(\widehat{N}, \widehat{F})$ will be denoted by $\Psi^*_{\rtimes}(\widehat{N}, \widehat{F})$. An element $P \in \Psi^*_\pi(\widehat{N}, \widehat{F})$ has compact $\Gamma$-support if its Schwartz kernel has compact support in $(\widehat{N} \times_\pi \widehat{N})/\Gamma$: we denote *the algebra* of $\Gamma$-equivariant pseudodifferential operators of compact $\Gamma$-support by $\Psi^*_{\rtimes,c}(\widehat{N}, \widehat{F})$. For example

$$(D_0(\theta))_{\theta \in T} \in \mathrm{Diff}^1_{\rtimes}(\widehat{N}, \widehat{F}) \subset \Psi^1_{\rtimes,c}(\widehat{N}, \widehat{F}).$$

It is important to understand what Hypothesis A implies for the $C^0(T) \rtimes_r \Gamma$-linear operator $\mathcal{D}_0$ induced by the family $(D_0(\theta))$, acting on $L^2_{C(T) \rtimes_r \Gamma}$.

**Proposition 1.**
*1] The operator $\mathcal{D}_0$ defines a regular unbounded operator on the $C^0(T) \rtimes_r \Gamma$-Hilbert module $L^2_{C^0(T) \rtimes_r \Gamma}$.*
*2] If Hypothesis A holds then this operator is $L^2_{C^0(T) \rtimes_r \Gamma}$-invertible, with inverse induced by the $\Gamma$-equivariant family of operators $\{D_0(\theta))^{-1}\}_{\theta \in T}$.*
*3] One can write $D_0(\theta)^{-1} = A(\theta) + R(\theta)$ with $\{A(\theta)\}_{\theta \in T} \in \Psi^1_{\rtimes,c}(\widehat{N}, \widehat{F})$ and $\{R(\theta)\}_{\theta \in T} \in \Psi^{-\infty}_{\rtimes}(\widehat{N}, \widehat{F})$. Moreover $\{R(\theta)\}_{\theta \in T}$ extends to an operator*

$$\mathcal{R} \in \mathbb{B}(H^m_{C^0(T) \rtimes_r \Gamma}(\widehat{N}, \widehat{F}), H^k_{C^0(T) \rtimes_r \Gamma}(\widehat{N}, \widehat{F}))$$

*$\forall k, m \in \mathbb{Z}$.*

*Proof.* 1] The fact that $\mathcal{D}_0$ defines a regular unbounded operator is standard.
2] Let us prove that $\mathcal{D}_0$ is $L^2_{C(T) \rtimes_r \Gamma}$-invertible. Since $\mathcal{D}_0$ is regular, we may consider

$$\frac{\mathcal{D}_0}{\sqrt{\mathrm{Id} + \mathcal{D}_0^2}} \in \mathbb{B}(L^2_{C^0(T) \rtimes_r \Gamma}(\widehat{N}, \widehat{F}))$$

instead of $\mathcal{D}_0$. For each $\theta \in T$ let $H_\theta$ denote the Hilbert space $H_\theta = L^2(\pi^{-1}\theta, \widehat{F}|_{\pi^{-1}(\theta)})$. The collection $\mathcal{H} = (H_\theta)_{\theta\in T}$ together with $C^0_c(\widehat{N};\widehat{F})$ defines a continuous field of Hilbert spaces over $T$. The space $\mathrm{End}_\Gamma(\mathcal{H})$ of $\Gamma-$equivariant bounded measurable fields of operators $A = (A_\theta)_{\theta\in T}$ $(A_\theta \in H_\theta)$ is a $C^\star-$algebra where the norm is given by:

$$||A|| = \sup\{||A_\theta||,\ \theta \in T\}.$$

Moreover, one has a natural injective morphism of $C^\star-$algebras:

$$J : \mathbb{B}(L^2_{C(T)\rtimes_r\Gamma}(\widehat{N},\widehat{F})) \to \mathrm{End}_\Gamma(H).$$

So for any $\tilde{A} \in \mathbb{B}(L^2_{C(T)\rtimes_r\Gamma}(\widehat{N},\widehat{F}))$ the spectrum of $\tilde{A}$ coincides with the one of $J(\tilde{A})$. Since Hypothesis A means that $J(\frac{\mathcal{D}_0}{\sqrt{\mathrm{Id}+\mathcal{D}_0^2}})$ is invertible we get immediately that $\mathcal{D}_0$ is $L^2_{C(T)\rtimes_r\Gamma}$-invertible.
3] The proof is standard and left to the reader.

□

## 3. Étale groupoids and the $b$-calculus

### 3.1. The fibre-$b$-stretched product.

Recall that we are given a $\Gamma$-equivariant fibration $Z \to \widehat{M} \to T$, with $Z$ a fixed manifold with boundary: we follow Section 2 for the notation adopted here. The boundary of the model fiber $Z$ will decompose as a disjoint union of connected components:

$$\partial Z = \sqcup_{\alpha\in A} W^\alpha$$

Thus for each $\theta \in T$ the fiber $\pi^{-1}(\theta) := \widehat{M}_\theta$ has a boundary

$$\partial\widehat{M}_\theta = \sqcup_{\alpha\in A} W^\alpha_\theta$$

with $W^\alpha_\theta$ diffeomorphic to $W^\alpha$. We consider

$$\widehat{W}^\alpha := \cup_{\theta\in T}(W^\alpha_\theta)\,.$$

This is a connected component of $\partial\widehat{M}$ (recall that $T$ is assumed to be connected).

We shall now define the space that will carry the Schwartz kernels of the operators we are interested in. Let

$$\widehat{M} \times_\pi \widehat{M} = \{(p,p') \in \widehat{M} \times \widehat{M} \,|\, \pi(p) = \pi(p')\}$$

be the fiber product of $\widehat{M}$ with itself. This is a fibration over $T$, with fiber over $\theta$ equal to $\widehat{M}_\theta \times \widehat{M}_\theta$. Consider

$$B = \cup_\theta(\sqcup_{\alpha\in A} W^\alpha_\theta \times W^\alpha_\theta)$$

**Definition 2.** *The b-stretched product of the* $\Gamma$*-equivariant fibration* $\widehat{M} \to T$ *is, by definition, the blow-up of* $\widehat{M} \times_\pi \widehat{M}$ *along* $B$*:*

$$[\widehat{M} \times_\pi \widehat{M}; B] \tag{5}$$

Notice that the $b$-stretched product of the $\Gamma$-equivariant fibration $\widehat{M} \to T$ is exactly the fibre-$b$-stretched product considered in [27], but with a non-compact fibre. On the space $[\widehat{M} \times_\pi \widehat{M}; B]$ there is a well defined $\Gamma$-action obtained by lifting the diagonal action of $\Gamma$ on the fibration $\widehat{M} \times_\pi \widehat{M}$.

*For more on the b-stretched product and on what follows the reader is invited to consult the basic reference* [26]*.*

### 3.2. The $b$-calculus $\Psi^{*,\delta}_{b,\rtimes}(\widehat{M}, \widehat{E})$.

The small fiber-$b$-calculus $\Psi^*_{b,\pi}(\widehat{M}, \widehat{E})$ is defined, exactly as in [27], in terms of the fiber $b$-stretched product $[\widehat{M} \times_\pi \widehat{M}; B]$ and the lifted fiber diagonal $\Delta_{b,\pi}$. We define the small $\Gamma$-*invariant fibre-b-calculus* as

$$\Psi^*_{b,\rtimes}(\widehat{M}, \widehat{E}) := \{P \in \Psi^*_{b,\pi}(\widehat{M}, \widehat{E}) \,|\, R^*_g \circ P = P \circ R^*_g \;\forall\, g \in \Gamma\} \tag{6}$$

The Schwartz kernel of an operator in $\Psi^*_{b,\rtimes}(\widehat{M}, \widehat{E})$ is a $\Gamma$-invariant distribution on $[\widehat{M} \times_\pi \widehat{M}, B]$. The subspace of operators with Schwartz kernel *compactly supported* in $[\widehat{M} \times_\pi \widehat{M}, B]/\Gamma$ will be denoted by $\Psi^*_{b,\rtimes,c}(\widehat{M}, \widehat{E})$; as in the closed case we shall say that elements in the latter space have compact $\Gamma$-support. The subspace of $b$-differential operators is denoted $\mathrm{Diff}^*_{b,\rtimes}(\widehat{M}, \widehat{E})$. We have already remarked that the family of Dirac operators $(D(\theta))_{\theta \in T}$ introduced in Section 2, defines an element in $\mathrm{Diff}^1_{b,\rtimes}(\widehat{M}, \widehat{E}) \subset \Psi^1_{b,\rtimes,c}(\widehat{M}, \widehat{E})$.

Let us fix a $\Gamma$-invariant trivialization $\nu$ of the positive normal bundle $N_+\partial\widehat{M}$ to the boundary of $\partial\widehat{M}$. Let $x \in C^\infty(\widehat{M})$ be a $\Gamma$-invariant boundary defining function such that $dx(\widehat{\nu}) = 1$. Let

$$\overline{N_+\partial\widehat{M}} \simeq \partial\widehat{M} \times [-1, 1]$$

denote its fibre compactification. The notion of indicial operator and indicial family are as in [26]. The indicial operator of $P \in \Psi^m_{b,\rtimes}(\widehat{M}, \widehat{E})$, denoted $I(P)$, is the $\mathbb{R}^+$-invariant element in $\Psi^m_{b,\rtimes}(\overline{N_+\partial\widehat{M}}, \widehat{E}_{|_{\partial\widehat{M}}})$ obtained by restricting the kernel of $P$ to the front face. The indicial operator defines a short exact sequence

$$0 \longrightarrow \rho_{\mathrm{bf}} \Psi^m_{b,\rtimes}(\widehat{M}, \widehat{E}) \longrightarrow \Psi^m_{b,\rtimes}(\widehat{M}, \widehat{E}) \xrightarrow{I(\,)} \Psi^m_{b,\rtimes,I}(\overline{N_+\partial\widehat{M}}, \widehat{E}_{|_{\partial\widehat{M}}}) \longrightarrow 0 \tag{7}$$

with the last space denoting the $\mathbb{R}^+$-invariant elements in $\Psi^m_{b,\rtimes}(\overline{N_+\partial\widehat{M}}, \widehat{E}_{|_{\partial\widehat{M}}})$. The indicial family of $P \in \Psi^m_{b,\rtimes}(\widehat{M}, \widehat{E})$ is obtained by fiber-Mellin transform from $I(P)$, once the trivialization $\nu$ has been fixed. It defines an entire map with values $\Gamma$-invariant pseudodifferential operators along the fibres of $\partial\widehat{M}$:

$$\mathbb{C} \ni z \to I_\nu(P, z) \in \Psi^m_{\rtimes}(\partial\widehat{M}, \widehat{E}_{|_{\partial\widehat{M}}})$$

where $\Psi^m_{\rtimes}(\partial\widehat{M}, \widehat{E}_{|_{\partial\widehat{M}}})$ denotes the set of $\Gamma-$equivariant families of pseudo-differential operators of order $m$ acting in the fibers of $\pi : \partial\widehat{M} \to T$.

Proceeding as in [27] (Appendix) and [17] (Section 7) we can also define the appropriate full calculi with bounds

$$\Psi^{*,\delta}_{b,\pi}(\widehat{M},\widehat{E}) := \Psi^{*}_{b,\pi}(\widehat{M},\widehat{E}) + \Psi^{-\infty,\delta}_{b,\pi}(\widehat{M},\widehat{E}) + \Psi^{-\infty,\delta}_{\pi}(\widehat{M},\widehat{E})$$

for $\delta > 0$. Here $\Psi^{-\infty,\delta}_{b,\pi}(\widehat{M},\widehat{E})$ is defined as in [17], right after formula (7.3), but with $[\widehat{M}\times_\pi \widehat{M}, B]$ replacing $\widetilde{M}^2_b$; similarly the residual space $\Psi^{-\infty,\delta}_{\pi}(\widehat{M},\widehat{E})$ is defined as

$$\rho^\delta_{lb}\rho^\delta_{rb} H^\infty_{b,\mathrm{loc}}(\widehat{M}\times_\pi \widehat{M}, \widehat{E}\boxtimes \widehat{E}^*). \tag{8}$$

**Remark.** We remark as in [27] (Appendix) that $\Psi^{-\infty,\delta}_{b,\pi}(\widehat{M},\widehat{E})$ can also be obtained by doubling $[\widehat{M}\times_\pi \widehat{M}, B]$ across the front face $bf$, thus obtaining $\mathcal{D}_{bf}[\widehat{M}\times_\pi \widehat{M}, B]$ and then declaring that $A\in \Psi^{-\infty,\delta}_{b,\pi}(\widehat{M},\widehat{E})$ if there exists $\epsilon>0$ and

$$\widetilde{A}\in \rho^{\delta+\epsilon}_{\mathrm{tot}} H^\infty_{b,\mathrm{loc}}\big(\mathcal{D}_{bf}[\widehat{M}\times_\pi \widehat{M}, B];(E\boxtimes E^*)_{\mathcal{D}}\big)$$

such that

$$\widetilde{A}|_{[\widehat{M}\times_\pi \widehat{M}, B]} = A\,.$$

In this definition we have used the obvious inclusion $[\widehat{M}\times_\pi \widehat{M}, B]\subset \mathcal{D}_{bf}[\widehat{M}\times_\pi \widehat{M}, B]$; moreover we have denoted by $\rho_{\mathrm{tot}}$ a total boundary defining function for $\mathcal{D}_{bf}[\widehat{M}\times_\pi \widehat{M}, B]$ and by $(E\boxtimes E^*)_{\mathcal{D}}$ the bundle obtaned by doubling $E\boxtimes E^*$ across $bf$.

The $\Gamma$-invariant elements in $\Psi^{*,\delta}_{b,\pi}(\widehat{M},\widehat{E})$ will be denoted by $\Psi^{*,\delta}_{b,\rtimes}(\widehat{M},\widehat{E})$; thus

$$\Psi^{*,\delta}_{b,\rtimes}(\widehat{M},\widehat{E}) := \Psi^{*}_{b,\rtimes}(\widehat{M},\widehat{E}) + \Psi^{-\infty,\delta}_{b,\rtimes}(\widehat{M},\widehat{E}) + \Psi^{-\infty,\delta}_{\rtimes}(\widehat{M},\widehat{E}) \tag{9}$$

The composition rules for these calculi are as in the Appendix of [27] (Theorem 4); notice that since the fibres $\widehat{M}_\theta$ are non-compact, for the composition of two operators $P,Q\in \Psi^{*,\delta}_{b,\rtimes}(\widehat{M},\widehat{E})$ to be well defined it is necessary to assume that one of them is compactly $\Gamma-$supported. Because of our condition on the support, elements in $\Psi^{*,\delta}_{b,\rtimes,c}(\widehat{M},\widehat{E})$ can be composed exactly as in Theorem 4 of ([27],Appendix).

The mapping properties of elements in $\Psi^{m}_{b,\rtimes}(\widehat{M},\widehat{E})$ are obtained by extending to the present context the arguments in [26]. First of all, by proceeding as in Lemma 1 one can check that an element in $\Psi^{m,\delta}_{b,\rtimes}(\widehat{M},\widehat{E})$ defines a $C^\infty_c(T\rtimes\Gamma)$-linear operator from $C^\infty_c(\widehat{M},\widehat{E})$ into $C^\infty(\widehat{M},\widehat{E})$. Moreover, the following Proposition (with proof as in [26]) holds

**Proposition 2.** *Let $P\in \Psi^{m,\delta}_{b,\rtimes,c}(\widehat{M},\widehat{E})$, $\delta>0$ Then for each $k\in\mathbb{Z}$, $P$ defines a bounded operator*

$$P: H^k_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M},\widehat{E}) \longrightarrow H^{k-m}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M},\widehat{E}) \tag{10}$$

*of Hilbert $C^0(T)\rtimes_r\Gamma$-modules.*

### 3.3. Elliptic elements and $b$-parametrices.

The goal of this subsection is to construct a parametrix associated to a $\Gamma$-equivariant family of odd Dirac operators $D \in \mathrm{Diff}^1_{b,\rtimes}(\widehat{M}, \widehat{E})$ under the assumption that the boundary family $D_0$ satisfies Hypothesis A of Section 2 with $\widehat{N} = \partial\widehat{M}$. The proof proceeds along the lines of the parametrix construction given in [26]. See also [27] and [17].

First notice that $D$ admits a symbolic parametrix $Q_\sigma \in \Psi^{-1}_{b,\rtimes,c}(\widehat{M}, \widehat{E})$, with $Q_\sigma$ odd; $Q_\sigma$ is obtained by proceeding as in [6]. If we write

$$Q = \begin{pmatrix} 0 & Q^- \\ Q^+ & 0 \end{pmatrix}$$

we have

$$Q^+ D^+ = \mathrm{Id} - S_{+,\sigma}\,, \qquad D^+ Q^+ = \mathrm{Id} - S_{-,\sigma} \tag{11}$$

with rests $S_{-,\sigma}$ and $S_{+,\sigma}$ in $\Psi^{-\infty}_{b,\rtimes,c}(\widehat{M}, \widehat{E})$. Hypothesis A implies the existence of the $L^2$-inverse of the indicial family of $D^+(\theta)$, $\forall \theta \in T$, i.e. the existence of the $\Gamma$-equivariant family of operators $\{(D_0(\theta) + i\lambda)^{-1}\}_{\theta \in T}$. We obtain in this way an element $(D_0 + i\lambda)^{-1} \in \Psi^{-1}_{\rtimes}(\partial\widehat{M}, \widehat{E}_{|_{\partial\widehat{M}}})$. By Proposition 1 we know that the family $(D_0 + i\lambda)^{-1}$ extends for each $\lambda \in \mathbb{R}$ to a bounded operator $(\mathcal{D}_0 + i\lambda)^{-1}$ on $L^2_{C(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{|_{\partial\widehat{M}}})$; moreover, $(D_0 + i\lambda)^{-1} = A_\lambda + B_\lambda$ where $A_\lambda \in \Psi^{-1}_{\rtimes,c}(\partial\widehat{M}, \widehat{E}_{|_{\partial\widehat{M}}})$ uniformly in $\lambda$ (i.e. the Schwartz kernel of $A_\lambda$ is included in a fixed compact $K \subset (\partial\widehat{M} \times_\pi \partial\widehat{M})/\Gamma$) and where $B_\lambda$ is given by a smooth Schwartz kernel and extends to a bounded operator $\mathcal{B}_\lambda$ from $H^k_{C(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{|_{\partial\widehat{M}}})$ into $H^l_{C(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{|_{\partial\widehat{M}}})$ for any $k, l \in \mathbb{Z}$. Let us consider the indicial family

$$\mathbb{R} \ni \lambda \to (D_0 + i\lambda)^{-1} \circ I(S_{-,\sigma}, \lambda) = A_\lambda \circ I(S_{-,\sigma}, \lambda) + B_\lambda \circ I(S_{-,\sigma}, \lambda)\,,$$

which is well defined since $S_{-,\sigma} \in \Psi^{-\infty}_{b,\rtimes,c}(\widehat{M}, \widehat{E})$. We can take the inverse Mellin transform of this indicial family and obtain an element in $\Psi^{-\infty,\delta}_{b,\rtimes,I}(\overline{N_+\partial\widehat{M}}, \widehat{E}_{|_{\partial\widehat{M}}})$; we now consider the operator $Q' \in \Psi^{-\infty,\delta}_{b,\rtimes}$ corresponding to this operator via the analogue of (7) for the calculus with bounds. The operator $Q^+ := Q^+_\sigma - Q'$ belongs to $\Psi^{-1,\delta}_{b,\rtimes}(\widehat{M}; \widehat{E}^+, \widehat{E}^-)$ and defines a right inverse modulo $\rho_{\mathrm{bf}}\Psi^{-\infty,\delta}_{b,\rtimes}$; this we call a right $b$-parametrix. Notice that

$$\rho_{\mathrm{bf}}\Psi^{-\infty,\delta}_{b,\rtimes}(\widehat{M}, \widehat{E}) \subset \Psi^{-\infty,\delta}_{\rtimes}(\widehat{M}, \widehat{E}) \tag{12}$$

Thus $Q^-$ provides an inverse of $D^+$ modulo $\Psi^{-\infty,\delta}_{\rtimes}(\widehat{M}, \widehat{E})$. Standard arguments ([26], p. 185) show that this is also a left $b$-parametrix. Thus: if Hypothesis A holds, then there exists $Q^+ \in \Psi^{-1,\delta}_{b,\rtimes}$ such that

$$D^+ Q^- = \mathrm{Id} - S_-\,, \quad Q^- D^+ = \mathrm{Id} - S_+\,, \quad \text{with} \quad S_-, S_+ \in \Psi^{-\infty,\delta}_{\rtimes} \tag{13}$$

More information about $Q'$ can be found in the next subsection.

If $P \in \mathrm{Diff}^m_{b,\rtimes}(\widehat{M}, \widehat{E})$ is more generally an elliptic differential $\Gamma$-invariant family, then we can find a symbolic parametrix $Q_\sigma \in \Psi^{-m}_{b,\rtimes,c}(\widehat{M}, \widehat{E})$, see [7]. If, in addition, the indicial family of $P$, $I_\nu(P, \lambda)$, satisfies Hypothesis A uniformly in $\lambda$, then we can proceed as above and find an inverse modulo $\rho_{\mathrm{bf}}\Psi^{-\infty,\delta}_{b,\rtimes}(\widehat{M}, \widehat{E}) \subset \Psi^{-\infty,\delta}_{\rtimes}(\widehat{M}, \widehat{E})$.

*Summarizing*, we have shown that an elliptic element in $\mathrm{Diff}^m_{b,\rtimes}(\widehat{M},\widehat{E})$ with invertible indicial family, admits an inverse $Q \in \Psi^{-m,\delta}_{b,\rtimes}$ modulo elements in $\Psi^{-\infty,\delta}_{\rtimes}$:

$$PQ = \mathrm{Id} - R_0\,, \qquad QP = \mathrm{Id} - R_1\,, \quad \text{with} \quad R_0, R_1 \in \Psi^{-\infty,\delta}_{\rtimes} \tag{14}$$

3.4. **The $b$-index class.**

The pseudoinverse $Q^+ \in \Psi^{-m,\delta}_{b,\rtimes}$ associated to $D^+$ constructed in the previous subsection will *not* be, in general, of compact $\Gamma$-support. The problem comes from the correction term $Q'$, which involves the *inverse* of the indicial family. Indeed, the Schwartz kernel of $Q'$ in the case for example of Dirac operators, has the following expression in a neighbourhood of the front face and in projective coordinates $(s, y, y')$: (recall that $s = \frac{x}{x'}$)

$$K(Q')(x, s, y, y') = \phi(x) \int_{\mathbb{R}} s^{i\lambda} K(\,(D_0 + i\lambda)^{-1} \circ I(S_{\sigma,-}, \lambda)\,)(y, y') d\lambda \tag{15}$$

where $\phi$ is a given smooth function in $C^\infty([0,1];[0,1])$ such that $\phi(x) = 1$ for $0 \leq x \leq \frac{1}{2}$ and $\phi(x) = 0$ for $x \geq \frac{3}{4}$ (recall that $K(Q')$ vanishes identically outside a neighborhood of the front face). As a consequence, the rests $S_+, S_-$ (where $S_- = S_{\sigma,-} - D^+Q'$) are not of compact $\Gamma$-support. In particular, we cannot conclude that $S_+, S_-$ define $(C(T) \rtimes_r \Gamma)$-compact operators: more work is needed in order to show that this is indeed the case and that there is a well defined index class.
We state first the result and we devote the next subsection to its proof:

**Theorem 1.**
*1] Let $D = (D(\theta)_{\theta\in T})$ a $\Gamma$-invariant family of odd $\mathbb{Z}_2$-graded Dirac operators. If Hypothesis A holds for the boundary family, then $D^+ = (D^+(\theta))$ defines $\forall m \in \mathbb{N}^*$ a $C^0(T) \rtimes_r \Gamma$-linear bounded operator*

$$H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}^+) \longrightarrow H^{m-1}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}^-)$$

*which is invertible modulo $C^0(T) \rtimes_r \Gamma$-compacts.*
*2] There is a well defined index class $\mathrm{Ind}(D^+)$ in $K_0(C^0(T) \rtimes_r \Gamma)$.*

3.5. **Proof of Theorem 1.**

In order to simplify the notation we assume that $\widehat{E}$ is the product line bundle $\widehat{M}\times\mathbb{C} \to \widehat{M}$.

**Lemma 4.** *Let $Q = Q_\sigma - Q'$ be the parametrix constructed in the previous section, with $Q_\sigma$ of compact $\Gamma$-support. Then $\forall m \in \mathbb{N}$, $Q$ extends to a bounded operator from $H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ to $H^{m+1}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$.*

*Proof.* Since the Schwartz kernel of $Q_\sigma$ is of compact $\Gamma$-support, we know that $Q_\sigma$ extends to a bounded operator from $H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ to $H^{m+1}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$. Using Proposition 1 and the fact that the schwartz kernel of $I(S_{\sigma,-}, \lambda)$ is of compact $\Gamma$-support, one checks in a straightforward way that for each $\lambda \in \mathbb{R}$:

$$(D_0 + i\lambda)^{-1} \circ I(S_{\sigma,-}, \lambda)$$

is bounded from $H^m_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}^-_{|\partial\widehat{M}})$ to $H^{m+1}_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}^+_{|\partial\widehat{M}})$. Next we observe that near the boundary any element of $L^2_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ may be written as an element of

$L^2_b([0,1], L^2_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{\partial\widehat{M}})$ and that for $m \in \mathbb{N}^*$ any element of $H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ maybe written as an element of

$$L^2_b([0,1], H^m_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{\partial\widehat{M}})) + H^1_b([0,1], H^{m-1}_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{\partial\widehat{M}})).$$

Then using formula (15) one gets easily the Lemma. □

**Lemma 5.** *The Schwartz kernel of $S_-$ satisfies $K(S_-)(z,z') = \rho^{\epsilon}_{lb}\rho^{\epsilon}_{rb}K(R')(z,z')$ for a suitable $\epsilon > 0$ where $R'$ induces a bounded operator from $H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ to $H^{m+1}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E})$ for any $m \in \mathbb{N}$.*

*Proof.* Recall that $S_- = S_{\sigma,-} - D^+Q'$. Let $\psi(z,z')$ be a smooth real valued function on $\widehat{M}^2$ such that $\psi(z,z') = 1$ when both $x = x(z)$ and $x' = x(z')$ belong to $[0, \frac{1}{100}]$ and $\psi(z,z') = 0$ when $x$ or $x'$ is greater than 1. The required result is satisfied by the Schwartz kernel $(1-\psi(z,z'))K(S_-)(z,z')$. So we are left to prove the Lemma for $\psi(z,z')K(S_-)(z,z')$. With a small abuse of notation we identify $K(S_-)$ with its lift to the stretched product. Using formula (15) and employing projective coordinates $(x,s,y,y')$ on the stretched product we have

$$(\beta^*\psi)K(S_-)(x,s,y,y') = (\beta^*\psi)K(S_{\sigma,-})(x,s,y,y') -$$

$$(\beta^*\psi)x\partial_x\phi(x)\int_{\mathbb{R}} s^{i\lambda}K(\,(D_0+i\lambda)^{-1}\circ I(S_{\sigma,-},\lambda)\,)(y,y')d\lambda -$$

$$(\beta^*\psi)\phi(x)\int_{\mathbb{R}} s^{i\lambda}K(\,I(S_{\sigma,-},\lambda)\,)(y,y')d\lambda.$$

with $\beta$ equal to the blow-down map.

By construction, the indicial family of this operator vanishes identically. Using Proposition 1 one checks that for each real $\lambda$ the operators $(D_0 - i\lambda)^{-1}\circ I(S_{\sigma,-},\lambda)$ belong to $B(\,H^m_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{|\partial\widehat{M}}); H^{m+1}_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{|\partial\widehat{M}})\,)$. Following [26] (see Section 5.13 and Section 5.14 there) one then checks that there exists $\epsilon > 0$ such that one can write:

$$\psi(z,z')K(S_-)(z,z') = \rho^{\epsilon}_{lb}\rho^{\epsilon}_{rb}K(\beta_*R')(z,z')$$

with $(x,s) \to K(R')(x,s,y,y')$ induces an element of

$$H^{\infty}_b([0,1]^2; B(\,H^m_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{|\partial\widehat{M}}); H^{m+1}_{C^0(T)\rtimes_r\Gamma}(\partial\widehat{M}, \widehat{E}_{|\partial\widehat{M}})\,).$$

One then gets easily the Lemma. □

Finally, one can show, proceeding as in [17] (Lemma 11.2) that the following lemma holds:

**Lemma 6.** *For each $\epsilon > 0$, each $m \geq 0$ and each $k > 0$ the inclusion*

$$x^{\epsilon}H^{m+k}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}) \hookrightarrow H^m_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}) \tag{16}$$

*defines a $C^0(T)\rtimes_r\Gamma$-compact operator.*

**End of the proof of Theorem 1.** It is an immediate consequence of the previous lemmas. More precisely, we can consider

$$\mathrm{IND}_m(D^+) = [p - p_0] \in K_0(\mathbb{K}(H^m_{b,C^0(T)\rtimes_r\Gamma})) \tag{17}$$

where

$$p := \begin{pmatrix} S_+^2 & S_+(I+S_+)Q^+ \\ S_-D^+ & I-S_-^2 \end{pmatrix}, \quad p_0 := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

and where $Q^+$ is a parametrix as above with rests $S_\pm$. These index classes are all compatible through the natural isomorphisms

$$K_0(\mathbb{K}(H^m_{b,C^0(T)\rtimes_r\Gamma})) \simeq K_0(\mathbb{K}(H^\ell_{b,C^0(T)\rtimes_r\Gamma})).$$

Our index class $\mathrm{Ind}(D^+) \in K_0(C^0(T)\rtimes_r\Gamma)$ is defined as the image of any of these index classes under the natural isomorphism

$$K_0(\mathbb{K}(H^m_{b,C^0(T)\rtimes_r\Gamma})) \simeq K_0(C^0(T)\rtimes_r\Gamma)\,.$$

We observe that Connes' index class $\mathrm{IND}(D^+) \in K_0(C^*_r(M,\mathcal{F},E))$ is obtained by using the K-Theory isomorphism $K_0(\mathbb{K}(H^m_{b,C^0(T)\rtimes_r\Gamma})) \simeq K_0(C^*_r(M,\mathcal{F},E))$ induced by the isomorphism of algebras $\mathbb{K}(H^m_{b,C^0(T)\rtimes_r\Gamma}) \simeq C^*_r(M,\mathcal{F},E)$, see subsection 2.5.

**Remark.** It would be interesting to know whether the index class we have just constructed can be recovered using a pseudodifferential calculus *on* a suitable groupoid. Recall that for a single manifold with boundary, this is indeed possible. See for example [29], [33].

## 4. **Rapid decay**

### 4.1. **Virtually nilpotent groups and the rapidly decreasing algebra.**

In the rest of the paper we shall make the following hypothesis:

**Hypothesis B.** *The group $\Gamma$ is virtually nilpotent.*

Following [9] we shall now introduce a *dense subalgebra* $\mathcal{T}^\infty$ of $C^0(T)\rtimes_r\Gamma$ with

$$C^\infty(T)\rtimes\Gamma \subset \mathcal{T}^\infty \subset C^0(T)\rtimes_r\Gamma\,.$$

Let us fix a word-metric on $\Gamma$, $||\ ||$, and let

$$\mathcal{B}^\infty_\Gamma = \{f:\Gamma\to\mathbb{C} \mid \forall L\in\mathbb{N},\ \sup_{\gamma\in\Gamma}(1+||\gamma||)^L|f(\gamma)|<\infty\}$$

be the rapidly decreasing algebra in $C^*_r(\Gamma)$, a Fréchet algebra.

**Definition 3.** *We define $\mathcal{T}^\infty$ as*

$$\mathcal{T}^\infty := C^\infty(T,\mathcal{B}^\infty_\Gamma)\,.$$

Elements in $\mathcal{T}^\infty$ can be written as $\sum t_\gamma\gamma$ where the sum is now infinite but with the functions $t_\gamma$ satisfying the condition

$$\sup_{\theta\in T,\gamma\in\Gamma}\left[|t_\gamma|(\theta)(1+||\gamma||)^N\right]<\infty,\quad \forall N\in\mathbb{N} \tag{18}$$

together with all their covariant derivatives.

**Definition 4.** *Let $d$ be the distance on $\widehat{M}$ associated with the lift on $\widehat{M}$ of an ordinary metric $g$ on $M$. Fix $z_0\in\widehat{M}$. Then $C^\infty_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ is defined to be the subset of the elements $s\in C^\infty(\widehat{M},\widehat{E})$ such that for any $N\in\mathbb{N}$ and multi-index $\alpha$:*

$$\sup_{z\in\widehat{M}}\big(||\nabla^\alpha s(z)||(1+d(z,z_0))^N\big)<+\infty$$

*where $\nabla$ denotes a $\Gamma-$invariant covariant derivative. The space $C^\infty_{\mathcal{T}^\infty}(\partial\widehat{M},\widehat{E}_{|\partial\widehat{M}})$ is defined similarly.*

**Lemma 7.** *Both $C^\infty_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ and $C^\infty_{\mathcal{T}^\infty}(\partial\widehat{M},\widehat{E}_{|\partial\widehat{M}})$ are left $\mathcal{T}^\infty-$modules.*

*Proof.* We assume that $\widehat{E}$ is trivial. Let $s \in C^\infty_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ and let $t=\sum t_\gamma\gamma \in \mathcal{T}^\infty$. We define the module structure as follows:

$$(t\cdot s)(z) := \sum_{\gamma\in\Gamma} t_\gamma(\pi(z))(\gamma\cdot s)(z)\,.$$

We need to check that

$$\forall N\in\mathbb{N},\ \sup_{z\in\widehat{M}}|(t\cdot s)(z)|(1+d(z,z_0))^N\,<\,\infty \tag{19}$$

and similarly for the covariant derivatives. In order to check (19) we fix one $p\in\mathbb{N}$ such that $\sum(1+\|\gamma\|)^{-p}\,<\,\infty$. As $\Gamma$ is of polynomial growth such $p$ always exists. Let $N\in\mathbb{N}$, then fix $C_N>0$ such that:

$$\text{for } ||\gamma||\le\frac{d(z,z_0)}{2},\ |s(z\cdot\gamma)|\le C_N(1+d(z,z_0))^{-N-p},$$

$$\sup_{z\in\widehat{M}}|s(z)|\le C_N,\ \sup_{\theta\in T,\gamma\in\Gamma}|t_\gamma(\theta)|\,\le C_N$$

$$\forall\gamma\in\Gamma,\ \sup_{\theta\in T}|t_\gamma(\theta)|\le C_N(1+||\gamma||)^{-N-p}.$$

Then one has:

$$\sum_{\|\gamma\|\ge\frac{d(z,z_0)}{2}}|t_\gamma(\pi(z))||\gamma\cdot s(z)|(1+d(z,z_0))^N\,\le\,\sum_{\|\gamma\|\ge\frac{d(z,z_0)}{2}}C_N^2(1+\|\gamma\|)^{-p}\frac{(1+d(z,z_0))^N}{(1+\|\gamma\|)^N}$$

$$\le C_N^2 2^N\sum_{\gamma\in\Gamma}(1+\|\gamma\|)^{-p}<\infty$$

Moreover

$$\sum_{\|\gamma\|\le\frac{d(z,z_0)}{2}}|t_\gamma(\pi(z))||\gamma\cdot s(z)|(1+d(z,z_0))^N\,\le\,\sum_{\|\gamma\|\le\frac{d(z,z_0)}{2}}C_N^2(1+d(z,z_0))^{-p}\le$$

$$\sum_{\gamma\in\Gamma}C_N^2(\frac{1}{1+2\|\gamma\|})^p\,<\,\infty$$

This proves (19); for the covariant derivatives the argument is similar. The proof for $C^\infty_{\mathcal{T}^\infty}(\partial\widehat{M},\widehat{E}_{|\partial\widehat{M}})$ is the same. The lemma is proved. □

4.2. **The refined $b$-index class.**

Once the rapidly decreasing algebra $\mathcal{T}^\infty$ is fixed, we can define the rapidly decreasing $b$-calculus with bounds $\Psi^{*,\delta}_{b,\mathcal{T}^\infty}$ where

$$\Psi^{*,\delta}_{b,\rtimes,c} \subset \Psi^{*,\delta}_{b,\mathcal{T}^\infty} \subset \Psi^{*,\delta}_{b,\rtimes}.$$

Operators in $\Psi^{*,\delta}_{b,\mathcal{T}^\infty}$ are not of compact $\Gamma$-support but have precise asymptotic properties with respect to the $\Gamma$-action. The definition is somewhat technical and can be found in Appendix A (Section 12).

Operators in the rapidly decreasing calculus have natural mapping properties. First of all, an element in $\Psi^{*}_{b,\mathcal{T}^\infty}$ defines a $\mathcal{T}^\infty$-linear endomorphism of $C^\infty_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$. Moreover the following proposition holds:

**Proposition 3.** *If $\Gamma$ is virtually nilpotent and $P \in \Psi^{m}_{b,\mathcal{T}^\infty}$, then $P : H^{k+m}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M},\widehat{E}) \to H^{k}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M},\widehat{E})$ is bounded for each $k \in \mathbb{Z}$.*

Next we consider $K \in \Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E}) \cup \Psi^{-\infty,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$. Using the fact that $K$ is a $\Gamma-$equivariant family and that the estimates (49) in Appendix A hold, one checks easily that $K$ defines a $\mathcal{T}^\infty$-linear endomorphism of $H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$. Moreover, if $K \in \Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ then $K$ sends $H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ into $\rho^\epsilon H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$.

Finally, proceeding as in [26] (see Section 5.16) and as in [27] (Theorem 4), one proves the following composition rules.

**Proposition 4.** *For any $\epsilon > 0$, the spaces $\Psi^{-\infty,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ and $\Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ are two sided modules over the small calculus $\Psi^{*}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$; moreover $\Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ is a two sided-module over $\Psi^{-\infty,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$. Both $\Psi^{*}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ and $\Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ are algebras and we have:*

$$\Psi^{-\infty,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}) \circ \Psi^{-\infty,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}) = \Psi^{-\infty,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}) + \Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E}).$$

We shall need the following

**Proposition 5.** *Let $D^+ = (D^+(\theta))_{\theta\in T}$ a $\Gamma$-equivariant family with boundary operator $D_0 = (D_0(\theta))_{\theta\in T}$ satisfying Hypothesis A. Then there exists $Q^+ \in \Psi^{-1}_{b,\mathcal{T}^\infty}$ such that*

$$Q^+D^+ = \operatorname{Id} - S^+\,, \quad D^+Q^+ = \operatorname{Id} - S^-\,, \quad \textit{with} \quad S^\pm \in \Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty} \tag{20}$$

*Proof.* We just need to observe that the correction term in the construction of the true parametrix (see (15)) is rapidly decreasing. However this is an immediate consequence of finite propagation speed estimates, exactly as in [17] Proposition 1.5 and Proposition 9.1. $\square$

We are now in the position of defining a *refined* index class.

**Definition 5.** *Let $D^+ = (D^+(\theta))_{\theta\in T}$ be a $\Gamma$-equivariant family with boundary operator $D_0 = (D_0(\theta))_{\theta\in T}$ satisfying Hypothesis A. We define its index class in $K_0(\Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E}))$ by*

$$\operatorname{Ind}_{\mathcal{T}^\infty}(D^+) = [p - p_0] \in K_0(\Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})) \tag{21}$$

*where*

$$p := \begin{pmatrix} S_+^2 & S_+(I+S_+)Q^+ \\ S_- D^+ & I - S_-^2 \end{pmatrix}, \quad p_0 := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

**Remark.** There is an injection

$$J : \Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E}) \hookrightarrow \mathbb{K}(L^2_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M}, \widehat{E}))$$

and the index class defined in subsection 3.4 is simply the image of $\mathrm{Ind}_{\mathcal{T}^\infty}(D^+)$ under the morphism induced by $J$ in K-theory.

**Remark.** Let $\Gamma$ be an arbitrary finitely generated group. In the closed case Gorokhovsky and Lott defines a subalgebra $C^\infty(\Gamma, \mathcal{B}^\omega)$ of $C^0(T)\rtimes\Gamma$ by imposing an exponentially rapid decay on the coefficients $t_\gamma$ appearing in $\sum t_\gamma \gamma$. They can then construct the algebra of rapidly degreasing smoothing operators and prove the existence of a refined index class in the K-theory group of such an algebra.

One might wonder why this program cannot be implemented on a manifold with boundary. The reason is once again in the construction of the true parametrix and more precisely in formula (15), where the resolvent family of the boundary operator appears. One needs a notion of rapidly decreasing pseudodifferential operators containing the resolvent of the boundary operator, uniformly in $\lambda$. At the moment it is only for the groups of polynomial growth that we know how to define such a notion.

## 5. Noncommutative differential forms and higher eta invariants

### 5.1. Noncommutative differential forms.

We consider Lott's space of noncommutative differential forms $\Omega_*(\mathcal{B}^\infty_\Gamma)$ (see [23] or [17] page 22 for a definition). We then make the following

**Definition 6.** *For any non negative integers $k$ and $l$ we set:*

$$\widehat{\Omega}_{k,l}(T, \mathcal{B}^\infty_\Gamma) = \Omega^k(T)\widehat{\otimes}\Omega_l(\mathcal{B}^\infty_\Gamma)$$

*where $\Omega^k(T)$ denotes the Frechet space of smooth $k-$differential forms over $T$ and $\widehat{\otimes}$ denotes a complete projective graded tensor product. We also set:*

$$\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma) = \Pi_{k,l\in\mathbb{N}}\widehat{\Omega}_{k,l}(T, \mathcal{B}^\infty_\Gamma).$$

Elements in $\widehat{\Omega}_{k,l}(T, \mathcal{B}^\infty_\Gamma)$ are given by sums

$$\sum \alpha_{g_0,g_1,\cdots,g_\ell}(\theta) g_0 dg_1 \cdots dg_\ell$$

with $\alpha_{g_0,g_1,\cdots,g_\ell}(\theta)$ $k$-forms on $T$ which are, in addition, rapidly decreasing, together with their $\theta$-derivatives, with respect to $\|g_0\| + \cdots + \|g_k\|$. Using Lemma 18 and the fact that $\Gamma$ is virtually nilpotent one shows that $\widehat{\Omega}_0(T, \mathcal{B}^\infty_\Gamma)$ may be identified with $\mathcal{T}^\infty$.

We let an element $g$ of the group $\Gamma$ act on a form $\omega \in \Omega^*(T)$ by considering the pullback via the diffeomorphism defined by the action of $g$. We observe that $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)$ has a bi-module structure over $\mathcal{T}^\infty$; this is described by the following rules with $u$ denoting an element of $C^\infty(T)$ and $u\gamma \in \mathcal{T}^\infty$:

$$(\,\alpha\widehat{\otimes} g_0 dg_1 \cdots dg_l\,) \cdot u\gamma \;=\; (g_0 \cdots g_l)^*(u)\alpha\widehat{\otimes} g_0 dg_1 \cdots dg_l \gamma$$

$$u\gamma \cdot (\,\alpha \widehat{\otimes} g_0 dg_1 \cdots dg_l\,) \;=\; u\gamma^*(\alpha)\widehat{\otimes}\gamma g_0 dg_1 \cdots dg_l.$$

Moreover $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)$ is a Fréchet algebra whose product is defined by the following rules:

$$(22)\quad (\,\alpha\widehat{\otimes} g_0 dg_1 \cdots dg_l\,)\cdot(\,\beta\widehat{\otimes}\gamma_0 d\gamma_1\cdots d\gamma_p\,) = (-1)^{l.|\partial\beta|}\alpha\wedge(g_0\cdots g_l)^*(\beta)\,\widehat{\otimes} g_0 dg_1\cdots dg_l\gamma_0 d\gamma_1\cdots d\gamma_p$$

where $\alpha$ and $\beta$ are homogeneous differentials forms on $T$. We then introduce, following [9], the following space which will be the receptacle for the super-traces:

$$\overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma) = \widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)/\overline{[\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)\,,\,\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)]_t}\;\;;$$

this is a space of *noncommutative differential forms modulo the closure of the space of graded commutators.* We endow $\overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma)$ with the natural differential and we get a complex whose homology is denoted $\widehat{H}_*(T, \mathcal{B}^\infty_\Gamma)$.

This homology does not coincide with the topological noncommutative de Rham homology $\widehat{H}_*(\mathcal{T}^\infty)$ of $\mathcal{T}^\infty$ (see [12]), simply because $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)$ does not coincide with $\widehat{\Omega}_*(\mathcal{T}^\infty)$. But, as we shall see later, there is a natural map from $\widehat{\Omega}_*(\mathcal{T}^\infty)$ to $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)$ .

### 5.2. Operators with differential forms coefficients.

Following [9] (see also [17] page 26), we give the following

**Definition 7.** *We denote by*

$$\mathrm{Hom}^\infty_{\mathcal{T}^\infty}(\,C^\infty_{\mathcal{T}^\infty}(\widehat{N}, \widehat{F})\,;\,\widehat{\Omega}_{k,l}(T, \mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty} C^\infty_{\mathcal{T}^\infty}(\widehat{N}, \widehat{F})\,)$$

*the set of left $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)-$linear operator $K$ acting on $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty} C^\infty_{\mathcal{T}^\infty}(\widehat{N}, \widehat{F})$ and sending any $F \in C^\infty_{\mathcal{T}^\infty}(\widehat{N}, \widehat{F})$ to $KF$ defined for any $z \in \widehat{N}$ by:*

$$(KF)(z) = \sum_{g_1,\dots,g_l\in\Gamma} dg_1\cdots dg_l \int_{\pi^{-1}(z\cdot(g_1\cdots g_l)^{-1})} K_{g_1,\dots,g_l}(z,w)F(w)dVol_{\pi^{-1}(z\cdot(g_1\cdots g_l)^{-1})}(w)$$

*where*

$$K_{g_1,\dots,g_l} : (z,w)\in\widehat{N}\times\widehat{N} \to \bigwedge^k(T^*_{\pi(z)}T)\otimes \mathrm{Hom}(\widehat{F}_w\,,\,\widehat{F}_z)$$

*vanishes for $\pi(z)\cdot(g_1\cdots g_l)^{-1} \neq \pi(w)$ and defines a smooth section on*

$$\{(z,w)\in\widehat{N}\times\widehat{N} \mid \pi(z)\cdot(g_1\cdots g_l)^{-1} = \pi(w)\}.$$

*Moreover, we require the following decay estimates: for any fundamental domain $A$ of $\widehat{N}$, any integer $p > 1$ and any differential operator $P \in \mathrm{Op}\,(\widehat{N}\times_\pi\widehat{N})$*

$$\sup_{(z,w)\in\widehat{N}\times_\pi\widehat{N}} \left[d(zg_1\cdots g_l, A) + ||g_2|| + \cdots + ||g_l|| + d(w, Ag_1^{-1})\right]^p |P_{z,w}K_{g_1,\cdots,g_l}(zg_1\cdots g_l, w)|$$

*is finite.*

The algebra of differential operators $\mathrm{Op}\,(\widehat{N}\times_\pi\widehat{N})$ acting on $C^\infty(\widehat{N}\times_\pi\widehat{N}\,,\,\widehat{F}\boxtimes\widehat{F}^\star)$, which appears above, is obtained by considering manifolds with empty boundary in Definition 15 in Appendix A.

**Remarks.**
1) The $\mathcal{T}^\infty-$linearity is a strong assumption imposed on $K$ and on the $K_{g_1,\dots,g_l}$. It does not seem easy to characterize the $\mathcal{T}^\infty-$linearity by simple formulas involving the $K_{g_1,\dots,g_l}$ alone.

This fact of course already appears in the covering case ($T$ reduced to a point, see [22]) when one deals with operators with noncommutative differential forms coefficients. Let us simply observe that the condition on the support of the $K_{g_1,\dots,g_l}$ is an easy consequence of the rules (22) and of the $C^0(T)-$linearity of $K$.
2) Assuming $K$ to be $\mathcal{T}^\infty-$linear and proceeding as for Proposition 2.3 of [17], one checks easily that Definition 4 and the above decay estimate imply that $K$ indeed sends $C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})$ into $\widehat{\Omega}_{k,l}(T,\mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty} C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})$
3) For $k=l=0$, the operator $K$ induces a $C^0(T)\rtimes_r\Gamma-$compact operator of $L^2_{C^0(T)\rtimes_r\Gamma}(\widehat{N},\widehat{F})$.

We fix a function $\phi\in C^\infty_c(\widehat{N})$ which satisfies $\sum_{\gamma\in\Gamma}\gamma\cdot\phi=1$. Let us denote the unit element of $\Gamma$ by $e$. For any $\theta\in T$, one sets as in [9], $\mathrm{STR}_{<e>,Cl(1)}\,K(\theta)=$

$$\sum_{g_0,\dots,g_l\in\Gamma,\,g_0\cdots g_l=e}(dg_1\cdots dg_l)g_0\int_{\pi^{-1}(\theta)}\phi(w)\,\mathrm{Str}_{Cl(1)}\,K_{g_1,\cdots,g_l}(wg_0^{-1},w)dVol_{\pi^{-1}(\theta)}(w)\tag{23}$$

where the definition of the supertrace $\mathrm{Str}_{Cl(1)}$ acting on $(\widehat{F}_w)_\sigma=\widehat{F}_w\otimes\mathrm{Cl}(1)$ is recalled in [17] (page 21). Moreover, Definition 7 implies that $w\to K_{g_1,\cdots,g_l}(wg_0^{-1},w)$ is smooth on the fiber-diagonal in $\widehat{N}\times_\pi\widehat{N}$ and, under the natural identification between $\widehat{F}_w\simeq\widehat{F}_{wg_0^{-1}}$ (due to the $\Gamma$-invariance of $\widehat{F}$), that $K_{g_1,\cdots,g_l}(wg_0^{-1},w)$ belongs to $\bigwedge^k(T^*_{\pi(wg_0^{-1})}T)\otimes\mathrm{Hom}(\widehat{F}_w\,,\,\widehat{F}_w)$.

Notice that $\mathrm{STR}_{<e>,Cl(1)}\,K\in\overline{\overline{\Omega}}_*(T,\mathcal{B}^\infty_\Gamma)$ and that the definition of $\mathrm{STR}_{<e>,Cl(1)}\,K$ extends obviously to

$$\mathrm{Hom}^\infty_{\mathcal{T}^\infty}(\,C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})\,;\,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty}C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})\,)$$

which we define to be equal to

$$\Pi_{k,l\in\mathbb{N}}\,\mathrm{Hom}^\infty_{\mathcal{T}^\infty}(\,C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})\,;\,\widehat{\Omega}_{k,l}(T,\mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty}C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})\,).$$

**Notation.** We shall henceforth use the following notation:

$$\Psi^{-\infty}_{\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{N},\widehat{F})\;:=\;\mathrm{Hom}^\infty_{\mathcal{T}^\infty}(\,C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})\,;\,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty}C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})\,)\tag{24}$$

### 5.3. **The higher eta invariant.**

We fix a $\Gamma-$invariant horizontal distribution $T^H\widehat{N}$ such that

$$T\widehat{N}=T^H\widehat{N}\oplus T(\widehat{N}/T)$$

and consider the rescaled Bismut superconnection in the odd-dimensional context (our notation follows [27] or [17]):

$$\mathbb{B}^{Bismut}_s=\sigma s\mathcal{D}_0+\nabla^u-\sigma\frac{1}{4s}c(\tau),\;s\in\mathbb{R}^{+*}$$

where $\mathcal{D}_0$ is the $\mathcal{T}^\infty$-linear Dirac operator introduced above, $c(\tau)$ denotes the Clifford multiplication by the curvature 2-form $\tau$ of $T^H\widehat{N}$ and $\nabla^u$ is a certain unitary connection.

Then, following [9], we fix a function $h\in C^\infty_c(\widehat{M})$ such that $\sum_{\gamma\in\Gamma}\gamma\cdot h=1$ on $\widehat{N}$ and consider for each real $s>0$ the superconnection

$$\mathbb{B}_s=\mathbb{B}^{Bismut}_s+\sum_{\gamma\in\Gamma}d\gamma\otimes h\gamma^{-1}$$

which sends $C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})$ into $\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty} C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})$. We recall the following rule of computation valid for any $\omega\otimes f\in\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)\otimes_{\mathcal{T}^\infty} C^\infty_{\mathcal{T}^\infty}(\widehat{N},\widehat{F})$:

$$\mathbb{B}_s(\omega\otimes f)=d\omega\otimes f+(-1)^{\partial\omega}\omega\otimes\mathbb{B}_s(f).$$

In order to state a Lichnerowicz-type formula for $\mathbb{B}_s^2$ we introduce a few notations. Let $\{e_i\}_{i=1}^{2k-1}$ be a local orthonormal basis for $T\pi^{-1}(\theta)$ and let $\{c^i\}_{i=1}^{2k-1}$ be Clifford algebra generators, with $(c^i)^2=-1$. Let $\{\tau^\alpha\}_{\alpha=1}^{\dim T}$ be a local basis of $T^*T$ and let $E^\alpha$ denote exterior multiplication by $\tau^\alpha$.

We then have the odd-dimensional analogue of formula (4.11) of [9] :

$$\begin{aligned}(25)\qquad \mathbb{B}_s^2={}&s^2D_0^2+\frac{1}{4}\sum_{i,j}\tilde{F}_{i,j}(\widehat{V})[c^i,c^j]+\sum_{i,\alpha}\tilde{F}_{\alpha,i}(\widehat{V})\,E^\alpha c^i+\frac{1}{4}\sum_{\alpha,\beta}\tilde{F}_{\alpha,\beta}(\widehat{V})[E^\alpha,E^\beta]\\ &-s\sum_{\gamma\in\Gamma}d\gamma(\,c(d^{vert}h)+E(d^{hor}h)\,)\gamma^{-1}-\sum_{\gamma,\gamma'}(\gamma\gamma'\cdot h)(\gamma\cdot h)d\gamma d\gamma'(\gamma\gamma')^{-1}\end{aligned}$$

Notice that the $\Gamma-$invariance of the horizontal distribution $T^H\widehat{N}$ implies that $d\mathbb{B}_s/ds$ is $\mathcal{T}^\infty$-linear (i.e. "vertical"). Now we define $\exp(-\mathbb{B}_s^2)$ using Duhamel formula around $\mathcal{D}_0^2$:

$$e^{-\mathbb{B}_s^2}=e^{-s\mathcal{D}_0^2}+\int_0^1 e^{-u_1s^2\mathcal{D}_0^2}(\mathcal{D}_0^2-\mathbb{B}_s^2)e^{(-(1-u_1)s^2\mathcal{D}_0^2}du_1+$$

$$\int_0^1\int_0^{1-u_1}e^{-u_1s^2\mathcal{D}_0^2}(\mathcal{D}_0^2-\mathbb{B}_s^2)e^{-u_2s^2\mathcal{D}_0^2}(\mathcal{D}_0^2-\mathbb{B}_s^2)e^{-(1-u_1-u_2)s^2\mathcal{D}_0^2}du_2du_1+\dots$$

Now, since by Hypothesis A $\mathcal{D}_0$ is invertible, we may apply to $K(e^{-s^2D_0^2})(z,w)$ the finite propagation speed estimates (see [17] formula (2.14)). So, there exists $\delta>0$ such that for any $a,N\in\mathbb{N}$ one has:

$$\forall(z,w)\in\widehat{N}\times_\pi\widehat{N},\ \forall s\geq 1,\ |(D_0^ae^{-s^2D_0^2})|(z,w)\leq C(a,N)(1+d(z,w))^N\exp(-s^2\delta).$$

Of course, $K(e^{-s^2D_0^2})(z,w)=0$ for $\pi(z)\neq\pi(w)$. Therefore, for each $s>0$ $e^{-s^2\mathcal{D}_0^2}$, defines an element in $\Psi^{-\infty}_{\mathcal{T}^\infty}(\widehat{N}\,;\,\widehat{F})$

Using these estimates, one shows as in Section 3 of [17] that the operators

$$(26)\qquad e^{-\mathbb{B}_s^2}\ \text{ and }\ \frac{d\mathbb{B}_s}{ds}\,e^{-\mathbb{B}_s^2}\ \text{ both belongs to }\ \Psi^{-\infty}_{\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{N},\widehat{F})$$

uniformly with respect to $s\to+\infty$.

Let $\mathcal{R}$ be the rescaling operator on $\overline{\widehat{\Omega}}_*(T,\mathcal{B}^\infty_\Gamma)$ which multiplies by $(2i\pi)^{-p}$ a form of degree $2p-1$ or of degree $2p$. We observe that $\mathcal{R}$ commutes with the differential $d$ of $\overline{\widehat{\Omega}}_*(T,\mathcal{B}^\infty_\Gamma)$. We then introduce

$$\widetilde{\eta}_{<e>}(s)=\mathcal{R}(\mathrm{STR}_{<e>,Cl(1)}(\,\frac{d\mathbb{B}_s}{ds}\,e^{-\mathbb{B}_s^2}\,))\,.$$

**Proposition 6.** *Under Hypothesis A and B, the higher eta invariant*

$$\widetilde{\eta}_{<e>}=\frac{2}{\sqrt{\pi}}\int_0^{+\infty}\widetilde{\eta}_{<e>}(s)\,ds\ \in\ \overline{\widehat{\Omega}}_*(T,\mathcal{B}^\infty_\Gamma)$$

*is well defined.*

*Proof.* The large time behaviour is treated as in [17], using (26). Thus we just have to show the convergence of the integral at $s = 0$, but this is basically done as in the proof of Proposition 26 page 219 of [23], taking into account theorem 2 in [9]. □

**Remark.** In the covering case, the definition of higher eta invariant, and the proof of the convergence of the integral defining it, are due to Lott (see section 4 of [23]).

Proceeding as in [23] page 220, one proves the following:

**Proposition 7.** *Consider a smooth path of superconnections* $\mathbb{B}_s(r), 0 \leq r \leq 1$ *associated to a smooth path* $D_0(r)$ $(0 \leq r \leq 1)$ *of Dirac type operators satisfying Assumption A. We assume that* $\mathbb{B}(r)$ *is obtained by smoothly varying with respect to* $r$*, the metric on* $N$*, the* $\Gamma-$*invariant metric in the fibers, the* $\Gamma-$*invariant connection on* $\widehat{F}$*, the horizontal distribution* $T^H\widehat{N}$ *and the function* $h \in C^\infty_c(N)$*. Then one has the following variational formula for the higher eta invariant* $\widetilde{\eta}_{<e>}(r)$ *associated to* $\mathbb{B}_s(r)$ *as in Proposition 6:*

$$\frac{d\widetilde{\eta}_{<e>}(r)}{dr} = -\frac{2}{\sqrt{\pi}} \lim_{r\to 0^+} \mathcal{R}\ \mathrm{STR}_{Cl(1)}\left( \frac{d\mathbb{B}_s(r)}{dr}\, e^{-\mathbb{B}_s^2(r)} \right)$$

*in* $\overline{\overline{\widehat{\Omega}}}_*(T, \mathcal{B}^\infty_\Gamma)$ *modulo exact forms.*

*Remark* 1. The local index theory for étale groupoids developed in [9] (Section 4) insures the existence of the above limit.

## 6. The $b$-supertrace and the higher local index theorem

### 6.1. The $b$-supertrace of an element in $\Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E})$.

In the previous section we have introduced, in the closed case, the space of rapidly decreasing smoothing operators with differential form coefficients. We have denoted this space of operators by $\Psi^{-\infty}_{\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{N}\,;\,\widehat{F})$. Let us now pass to the case of manifolds with boundary and to the notion of rapidly decreasing $b$-smoothing operator with differential form coefficients. There are as usual three spaces:

$$\Psi^{-\infty}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E}), \quad \Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E}) \quad \Psi^{-\infty,\delta}_{\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E})$$

depending on the boundary behaviour of the coefficients $K_{g_1,\cdots,g_l}$ in the Schwartz kernel of the operator $K := \sum_{g_1,\cdots,g_l\in\Gamma} dg_1 \cdots dg_l\, K_{g_1,\cdots,g_l}$. We give the precise definition of these 3 spaces in Appendix B.

We now proceed to the definition of the $b$-supertrace and its properties. We fix a function $\phi \in C^\infty_c(\widehat{M})$ which is constant in the normal direction near the boundary and satisfies $\sum_{\gamma\in\Gamma} \gamma\cdot\phi = 1$.

As in Section 3 let us fix a $\Gamma-$invariant trivialization $\nu \in C^\infty(\partial\widehat{M}, N_+\partial\widehat{M})$ of the normal bundle and $x \in C^\infty(\widehat{M})$ a $\Gamma-$invariant boundary defining function for $\partial\widehat{M}$ such that $dx\cdot\nu = 1$ on $\partial\widehat{M}$. For any function $f \in C^\infty(\widehat{M} \times_\pi \widehat{M})$ we set for any $\theta \in T$,

$$^{\nu}\!\!\int_{\Delta_b(\theta)} \phi(w) f(w,w)\, dVol^b_{\pi^{-1}(\theta)}(w) :=$$

$$\lim_{\epsilon\to 0^+}[\int_{x>\epsilon}\phi(w)f(w,w)\,dVol^b_{\pi^{-1}(\theta)}(w)+\log\epsilon\int_{\partial\pi^{-1}(\theta)}\phi(w)f(w,w)\,dVol_{\partial\pi^{-1}(\theta)}(w)]$$

where $\Delta_b(\theta)$ denotes the lifted diagonal of $\pi^{-1}(\theta)$ in the associated $b-$streched product.

Now, for given $(k,l)\in\mathbb{N}$ we consider a particular element $K\in\Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E})$ thus with Schwartz kernel of the form

$$K(z,w)=\sum_{g_1,\cdots,g_l\in\Gamma}dg_1\cdots dg_l\,K_{g_1,\cdots,g_l}(z,w)$$

where for any $(z,w)\in\widehat{M}\times\widehat{M}$, $K_{g_1,\cdots,g_l}(z,w)\in\bigwedge^k(T^*_{\pi(z)}T)\otimes Hom(\widehat{E}_w\,,\,\widehat{E}_z)$ and vanishes for $\pi(z)(g_1\cdots g_l)^{-1}\neq\pi(w)$. Then we state the following

**Definition 8.** *For any $\theta\in T$ we set*

$$^bSTR_{<e>}(K)(\theta):=$$

$$\sum_{g_0,\cdots,g_l\in\Gamma,\,g_0\cdots g_l=e}(dg_1\cdots dg_l)g_0\,{}^{\nu}\!\!\int_{\Delta_b(\theta)}\phi(w)Str K_{g_1,\cdots,g_l}(wg_0^{-1},w)\,d\mathrm{Vol}^b_{Z_\theta}(w)$$

*Then $^bSTR_{<e>}(K)$ defines an element of $\overline{\widehat{\Omega}}_*(T,\mathcal{B}^\infty_\Gamma)$.*

Of course, this definition extends to any element of $\Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E})$.

**Remark.** The previous definition is also valid for any $K\in\Psi^{-\infty,\delta}_{\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E})$ but in this case the regularized integrals in the above definition are in fact ordinary integrals because the indicial family of $K$ vanishes identically.

Then, as in [17] (see Proposition 13.5) one gets the following commutator formula

**Proposition 8.** *For any $K,K'$ belonging to $\Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E})$ one has:*

$$^bSTR_{<e>}[K,K']\;=\;\frac{\sqrt{-1}}{2\pi}\int_{\mathbb{R}}STR\big(\frac{\partial}{\partial\lambda}I(K,\lambda)\circ I(K',\lambda)\big)d\lambda.$$

*If we replace $K$ by a differential operator $\in\mathrm{Diff}^1_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ and $K'$ by the composition of $K'$ with an element of the calculus with bounds $\Psi^{*,\delta}_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ then the same commutator formula is valid.*

Now we fix a $\Gamma-$invariant horizontal distribution $T^H\widehat{M}$ such that

$$^bT\widehat{M}=T^H\widehat{M}\oplus{}^bT(\widehat{M}/T)$$

and, as in [9] and [27] (section 9) we consider the Bismut superconnection

$$\mathbb{A}^{Bismut}_s=s\mathcal{D}+\nabla^u-\frac{1}{4s}c(\tau),\;\,s\in\mathbb{R}^{+*}$$

where $\mathcal{D}$ is the Dirac operator introduced in Lemma 1, $c(\tau)$ denotes the Clifford multiplication by the curvature 2-form $\tau$ of $T^H\widehat{M}$ and $\nabla^u$ is a certain unitary connection. Then, as in

[9] we fix a function $h \in C_c^\infty(\widehat{M})$ constant in the normal direction near the boundary such that $\sum_{\gamma\in\Gamma} \gamma \cdot h = 1$ on $\widehat{M}$ and consider for each real $s > 0$ the superconnection

$$\mathbb{A}_s = \mathbb{A}_s^{Bismut} + \sum_{\gamma\in\Gamma} d\gamma \otimes h\gamma^{-1}$$

which sends $C^\infty_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$ into $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma) \otimes_{\mathcal{T}^\infty} C^\infty_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$. By developing in a straightforward way a $\mathcal{T}^\infty$-$b-$heat calculus as in [17] (section 10) one sees easily that $e^{-s^2D^2} \in \Psi^{-\infty}_{b,\mathcal{T}^\infty}(\widehat{M}\,;\,\widehat{E})$ for any $s > 0$. Moreover, using a Duhamel expansion around $e^{-s^2D^2}$ one checks that for any $s > 0$

$$e^{-\mathbb{A}_s^2} \in \Psi^{-\infty}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{M}\,;\,\widehat{E})$$

Now as in [27] (section 10) we consider for any $s > 0$ the induced boundary superconnection

$$\mathbb{B}_s = s\sigma\mathcal{D}_0 + \nabla^u_\partial - \frac{\sigma c(\partial\tau)}{4s} + \sum_{\gamma\in\Gamma} d\gamma \otimes h\gamma^{-1}$$

with $D_0$ the boundary Dirac operator of $D$. Then according to Proposition 6 the higher eta invariant

$$\widetilde{\eta}_{<e>} = \frac{2}{\sqrt{\pi}}\mathcal{R}\int_0^{+\infty} STR_{<e>,Cl(1)}(\frac{d\mathbb{B}_s}{ds}e^{-\mathbb{B}_s^2})\,ds \in \overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma)$$

is well defined. Next we recall from Section 2 of [9] the definition of the following connection $\nabla^{can}$.

$$\forall f \in C_c^\infty(\widehat{M}),\ \nabla^{can} f = d^{\widehat{M}}\, f \oplus \sum_{\gamma\in\Gamma} d\gamma \otimes h(\gamma^{-1}\cdot f).$$

Then $(\nabla^{can})^2$ acts on $C_c^\infty(\widehat{M})$ as left multiplication by a $2-$form $\Theta$ which commutes with $C_c^\infty(\widehat{M}) \rtimes \Gamma$. Explicitly,

$$\Theta = \sum_{\gamma\in\Gamma} d^{\widehat{M}}(\gamma\cdot h)d\gamma\gamma^{-1} - \sum_{\gamma,\gamma'\in\Gamma}(\gamma\gamma'\cdot h)(\gamma\cdot h)d\gamma d\gamma'(\gamma\gamma')^{-1}.$$

Then put $\mathrm{ch}(\nabla^{can}) = e^{-\frac{\Theta}{2\pi i}}$, this Chern character lies in

$$\Pi_{k=0}^{\infty} \oplus_{l=0}^{+\infty} \Omega^{k-l}(\widehat{M})\hat{\otimes}\Omega^{k+l}(\mathbb{C}\Gamma). \tag{27}$$

Observe that for any function $\psi \in C_0^\infty(\widehat{M}, \mathbb{C})$, we may write

$$\psi\, e^{-\frac{\Theta}{2\pi i}} = \sum_{k\in\mathbb{N}} \omega_k$$

where each $\omega_k$ belongs to the algebraic tensor product $\Omega^*(\widehat{M})\hat{\otimes}\Omega^*(\mathbb{C}\Gamma)$ and is of total degree $k$.

## 6.2. **The short time limit.**

Now we may state the higher local index theorem whose proof proceeds along the line of the proofs of Theorem 2 of [9], ( see also Theorem 13.6 of [17] for the $b-$part ).

**Theorem 2.**

$$\lim_{s\to 0^+} \mathcal{R}({}^b STR_{<e>}\, e^{-\mathbb{A}_s^2}) \;=\; \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) \;\in \overline{\widehat{\Omega}}_*(T, \mathcal{B}_\Gamma^\infty)$$

*where $Z$ denotes the typical fiber of $\pi : \widehat{M} \to T$ and where we recall that $\phi \in C_c^\infty(\widehat{M})$ denotes a function which is constant in the normal direction near the boundary and such that $\sum_{\gamma\in\Gamma} \gamma\cdot\phi = 1$*

Recall now that $M$ is endowed with a foliation $\mathcal{F}$ whose leaves are the images by the covering map $p : \widehat{M} \to M$ of the fibers of $\pi : \widehat{M} \to T$. Then, let $\Phi$ be a closed graded trace of degree $n$ on the graded algebraic tensor product $\sum_{k+l=n} \Omega^k(T)\otimes\Omega^l(\mathbb{C}\Gamma)$. By proposition 2 of [9] we may write $\Phi = \sum_{k+l=n} \tau_{k,l}$ where $\tau_{k,l} \in C_k(T)\otimes C^l(\Gamma)$ ($C_k(T)$ denoting the set of currents of degree $k$); we shall assume that $\Phi$ extends as a closed graded trace on $\widehat{\Omega}_\star(T, \mathcal{B}_\Gamma^\infty)$. We shall now show that associated to $\nabla^{can}$ and $\Phi$ there is a well defined current $\omega_\Phi$ on $M$. First, by pairing with respect to the $\Gamma-$variables the Chern character associated to $\nabla^{can}$ and $\Phi$, we obtain an element

$$< \mathrm{ch}\,\nabla^{can}\,;\, \Phi >_\Gamma \in\; \Omega^*(\widehat{M}) \otimes C_*(T)\,;$$

then since $\pi : \widehat{M} \to T$ is a fibration we can associate to any $\omega\otimes\beta \in \Omega^*(\widehat{M})\otimes C_*(T)$ an element $\omega\otimes\pi^*(\beta)$ of $\Omega^*(\widehat{M})\otimes C_*(\widehat{M})$. Observe now that $C_*(\widehat{M})$ is a module over $\Omega^*(\widehat{M})$. Summarizing we can define

$$A : \Omega^*(\widehat{M}) \otimes C_*(T) \to C_*(\widehat{M}).$$

by $A(\omega\otimes\beta)(\lambda) = \pi^*(\beta)(\lambda\wedge\omega)$. Thus $A(< \mathrm{ch}\,\nabla^{can}\,;\,\Phi >_\Gamma)$ is well defined as a current on $\widehat{M}$ and it is, moreover, $\Gamma-$invariant. We denote by $\omega_\Phi$ the induced current on $M$. Recall that $p$ denotes the covering map $p : \widehat{M} \to M$. Thus, *by the very definition of* $\omega_\Phi$, one has for each $\alpha \in \Omega^*(M)$:

$$< \alpha\,;\,\omega_\Phi >=< \phi\, p^*(\alpha)\,;\, A(< \mathrm{ch}\,\nabla^{can}\,;\,\Phi >_\Gamma) >;$$

we refer to the proof of formula (31) below for the independence of the right hand side of the above formula on the choice of $\phi$ satisfying $\sum_{\gamma\in\Gamma}\gamma\cdot\phi = 1$.

Summarizing, we may state the following corollary:

**Corollary 1.** *Let $\Phi$ be a closed graded trace on $\widehat{\Omega}_*(T, \mathcal{B}_\Gamma^\infty)$ concentrated in the trivial conjugacy class. Then there is a current $\omega_\Phi$ on $M$ such that the following formula holds:*

$$< \int_Z \phi\,\widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can})\,;\,\Phi >=< \widehat{A}(T\mathcal{F})\,\mathrm{ch}(\nabla^V)\,,\,\omega_\Phi >$$

*where on the right hand side the pairing is the one between currents and differential forms on $M$.*

*Proof.* We write:

$$< \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\, \mathrm{ch}(\nabla^{\widehat{V}})\, \mathrm{ch}(\nabla^{can})\,;\, \Phi >=< \phi\, \widehat{A}\,(\nabla^{TZ})\, \mathrm{ch}(\nabla^{\widehat{V}})\,;\, A(< \mathrm{ch}\nabla^{can}\,;\, \Phi >_\Gamma)\; >$$

where on the right hand side the pairing is the one between currents and differential forms on $\widehat{M}$. Next we observe that $\widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})$ is a $\Gamma-$invariant differential form on $\widehat{M}$, it is then the pull back by the covering map $\widehat{M} \to M$ of $\widehat{A}(T\mathcal{F})\,\mathrm{ch}(\nabla^V)$. We then get immediately by definition of $\omega_\Phi$:

$$< \widehat{A}(T\mathcal{F})\, \mathrm{ch}(\nabla^V)\,,\, \omega_\Phi > = < \phi\, \widehat{A}\,(\nabla^{TZ})\, \mathrm{ch}(\nabla^{\widehat{V}})\,;\, A(< \mathrm{ch}\nabla^{can}\,;\, \Phi >_\Gamma)\; > .$$

The Corollary is thus proved. □

**Example 1 .** Let us describe $\omega_\Phi$ when $T = \{pt\}$. Then we are dealing with a single Galois covering $\Gamma \to \widetilde{M} \to M$, with $M$ orientable. If $\Phi$ is a graded closed trace on $\Omega_*(\mathcal{B}^\infty_\Gamma)$ and $\lambda$ is a smooth form on $M$ then

$$\forall \lambda \in \Omega^*(M),\ \omega_\Phi(\lambda) = \int_M \lambda \wedge < \omega, \Phi >$$

with $< \omega, \Phi >$ equal to a smooth differential form on $M$, see [23]. Therefore, in this case, $\omega_\Phi$ is simply the integration against $< \omega, \Phi > \in \Omega^*(M)$.

**Example 2 .** We now describe $\omega_\Phi$ in another simple example. Assume that the $\Gamma-$equivariant fibration $\pi$ is trivial, $\pi : \widehat{M} = \widetilde{M} \times T \mapsto T$ and that the quotient $\widetilde{M}/\Gamma$ is a smooth compact manifold with boundary. Consider a $\Gamma-$invariant measure $\mu$ on $T$ (since $\Gamma$ is of polynomial growth such a measure always exists). We define $\Phi$ by:

$$\forall\, \Xi \;=\; \sum_{\gamma \in \Gamma} \alpha_\gamma(\theta)\, \gamma \;\in \mathcal{T}^\infty \;=\; \widehat{\Omega}_0(T, \mathcal{B}^\infty), \quad \Phi(\Xi) = \mu(\alpha_e). \tag{28}$$

Thus

$$\Phi = \mu \otimes \delta_e \tag{29}$$

with $\delta_e$ the trivial cyclic 0-cocycle on $\mathcal{B}^\infty_\Gamma$. Since $\mathrm{ch}\,\nabla^{can}$ belongs to the space (27) and the component of $\mathrm{ch}\,\nabla^{can}$ living in $\Omega^0(\widehat{M})\widehat{\otimes}\Omega^0(\mathbb{C}\Gamma)$ is identically equal to 1, one checks easily that $A(< \mathrm{ch}\,\nabla^{can}\,;\, \Phi >_\Gamma) \;=\; \mu \circ \pi_*$ where $\pi_*$ denotes the integration in the fibers. Thus $\omega_\Phi(\alpha) = \mu(\pi_*(\phi p^*(\alpha)))$. Recall now that the fibers of $\pi : \widehat{M} = \widetilde{M} \times T \to T$ induce a foliation $\mathcal{F}$ on $M = \frac{\widetilde{M} \times T}{\Gamma}$ and that $\mu$ induces an invariant transverse measure, still denotes $\mu$, on $(M, \mathcal{F})$. Let $\phi$ be any function belonging to $C^\infty_c(\widehat{M}, [0,1])$ such that $\sum_{\gamma\in\Gamma} \gamma \cdot \phi = 1$ on $\widehat{M}$. Then the Ruelle-Sullivan current $RS$ on $(M, \mathcal{F})$ associated to the transverse measure $\mu$ is given by:

$$\forall \alpha \in C^\infty(M, \wedge^* T^* M), \quad RS(\alpha) = \mu(\int_{\widetilde{M}} p^*(\alpha)\phi)$$

where $p : \widehat{M} \to M$ denotes the covering map. Summarizing, in this particular case,

$$\omega_\Phi = RS\,. \tag{30}$$

**Remark.** Notice that if $\psi$ is any other function belonging to $C^\infty_c(\widehat{M}, [0,1])$ such that $\sum_{\gamma\in\Gamma} \gamma \cdot \psi = 1$ on $\widehat{M}$ then $\mu(\int_{\widetilde{M}} p^*(\alpha)\phi) = \mu(\int_{\widetilde{M}} p^*(\alpha)\psi)$. Indeed, using the fact that $p^*(\alpha)$

is $\Gamma-$invariant, that $\Gamma$ acts by isometries on $\widetilde{M}$ (by fixing a $\Gamma$-invariant metric) and that $\mu$ is $\Gamma$-invariant, one simply writes

$$(31)\qquad \mu(\int_{\widetilde{M}} p^*(\alpha)\phi) = \mu(\int_{\widetilde{M}} p^*(\alpha)\,\phi(\sum_{\gamma\in\Gamma}\gamma\cdot\psi)) = \mu(\sum_{\gamma\in\Gamma}\int_{\widetilde{M}} p^*(\alpha)\,\phi\,\gamma\cdot\psi) =$$

$$\mu(\sum_{\gamma\in\Gamma}\int_{\widetilde{M}} p^*(\alpha)\,\psi\,\gamma^{-1}\cdot\phi) = \mu(\int_{\widetilde{M}}(p^*(\alpha)\,\psi\sum_{\gamma\in\Gamma}\gamma^{-1}\cdot\phi\,)) \;=\; \mu(\int_{\widetilde{M}} p^*(\alpha)\psi).$$

**End of remark.**

Now, we are going to interpret $< \widetilde{\eta}_{<e>};\Phi >$ as an $L^2-$type invariant. To this aim we consider first a closed odd-dimensional Galois covering $\widetilde{N}\to N$ and the equivariant fibration $\pi:\widetilde{N}\times T\to T$. We shall later apply our remarks to $\widetilde{N}\times T = \partial\widetilde{M}\times T$. We still consider on $T$ a $\Gamma$-invariant measure $\mu$. First we construct a $L^2$-foliated trace on the space of $\Gamma-$equivariant families of smoothing operators acting in the fibers of $\pi:\widehat{N}\to T$, with $\widehat{N}=\widetilde{N}\times T$.

**Definition 9.** *Let $K=(K_\theta)_{\theta\in T}$ be a $\Gamma-$equivariant family of smoothing operators acting in the fibers of $\pi:\widehat{N}=\widetilde{N}\times T\to T$ on the sections of a equivariant vector bundle $\widehat{F}$. Then $tr\,K_\theta(z,z)\,dVol_{\pi^{-1}(\theta)}$ defines a longitudinal measure, denoted $tr\,K_\theta(y,y)\,dVol_{\pi^{-1}(\theta)}(y)$, on the leaves of the foliation $\mathcal{F}$ of $N$ induced by the image under the covering map $p:\widehat{N}\to N$ of the fibers $\widetilde{N}\times\{\theta\}$, $\theta\in T$. Denote still by $\mu$ the invariant transverse measure induced by $\mu$ on $(N,\mathcal{F})$. The $L^2$-$\mu$-trace of $K=(K_\theta)_{\theta\in T}$, denoted $\mathrm{Tr}_{(2),\mu}\,K$, is by definition the integration of the longitudinal measure $tr\,K_\theta(y,y)\,dVol_{\pi^{-1}(\theta)}(y)$ against the transverse measure $\mu$.*

**Lemma 8.** *For the $L^2$-$\mu$-trace of $K=(K_\theta)_{\theta\in T}$ the following formula holds:*

$$\mathrm{Tr}_{(2),\mu}\,K = \int_T(\int \phi\,tr\,K_\theta(z,z)\,dVol_{\pi^{-1}(\theta)})d\mu(\theta).$$

*Proof.* Let $F$ be a fundamental domain of $\widetilde{N}$ for the $\Gamma-$action. We fix a point $m_0\in\widetilde{N}$ and consider the transversal $\mathcal{T}_0$ of $\mathcal{F}$ in $N$ induced by the immersion of $\{m_0\}\times T$ in $\widehat{N}$. For any $(z,\theta)\in\widehat{N}\setminus(\,(\cup_{\gamma\in\Gamma}\partial F\cdot\gamma)\times T$ we consider the unique $\gamma\in\Gamma$ such that $z\cdot\gamma\in F\setminus\partial F$ and set $\chi((z,\theta)\cdot\Gamma)=(m_0,\theta\cdot\gamma)\in\mathcal{T}_0$. We define the restriction of $\chi$ to

$$\{(z,\theta)\cdot\Gamma,\ (z,\theta)\in(\,(\cup_{\gamma\in\Gamma}\partial F\cdot\gamma)\times T\}$$

to be any borel map such that $\chi((z,\theta)\cdot\Gamma)=(m_0,\widetilde{\theta})$ where there exists $\widetilde{z}\in\partial F$ such that $(z,\theta)\cdot\Gamma=(\widetilde{z},\widetilde{\theta})\cdot\Gamma$. We have thus defined a borel map $\chi: N=\frac{\widetilde{N}\times T}{\Gamma}\to\mathcal{T}_0$ such that for any $x\in N$, $\chi(x)$ belongs to the leaf of $x$. Notice that the following family

$$(\,tr\,K_\theta(z,z)\,dVol_{\pi^{-1}(\theta)}(z)\,)_{\theta\in T}$$

of $\Gamma-$invariant measures along each fiber of $\pi$ descends on $N$ and thus induces a longitudinal measure of $(N,\mathcal{F})$ that we denote by

$$tr\,K_\theta(y,y)\,dVol_{\partial\pi^{-1}(\theta)}(y).$$

Then, by definition, (see [30]):

$$\mathrm{Tr}_{(2),\mu}\,K = \int_T\int 1_{\chi^{-1}(\{(m_0,\theta)\})}(y)tr\,K_\theta(y,y)\,dVol_{\pi^{-1}(\theta)}(y)d\mu(\theta).$$

Using the fact that $\cup_{\gamma\in\Gamma}(\,\partial F\cdot\gamma\,)$ is of measure zero in $\widetilde{N}$, we observe that this last integral is equal to

$$\int_T\int 1_F(z) tr\, K_\theta(z,z)\, dVol_{\pi^{-1}(\theta)}(z) d\mu(\theta).$$

Since $\sum_{\gamma\in\Gamma} 1_F\cdot\gamma = 1$ on $\widetilde{N}$, we can use the same reasoning as in the proof of formula (31) in order to replace in the previous integral $1_F$ by $\phi$. This proves the Lemma. □

We *define* the $L^2$-foliated eta invariant of the family $D_0$ as:

$$\eta_{(2),\mu} := \frac{1}{\sqrt{\pi}}\int_0^\infty \eta_{(2),\mu}(t)dt \quad \text{with} \quad \eta_{(2),\mu}(t) = \mathrm{Tr}_{(2),\mu}(D_0\exp(-t^2 D_0^2)).$$

Now, recall that $\widetilde{\eta}_{<e>}$ and $\Phi$ are respectively defined in Proposition 6 and formula (28). Then, by applying Duhamel's formula one sees immediately that

$$\sqrt{\pi} < \widetilde{\eta}_{<e>};\Phi >= \int_0^{+\infty}\int_T(\int \phi\, tr\, K(D_0 e^{-t^2 D_0^2})(z,z)\, dVol_{\pi^{-1}(\theta)}\, dt\,) d\mu(\theta)$$

where of course $K(D_0 e^{-t^2 D_0^2})(\cdot,\cdot)$ denotes the (fiberwise) Schwartz kernel of $D_0 e^{-t^2 D_0^2}$. But then Lemma 8 shows that

$$< \widetilde{\eta}_{<e>};\Phi >= \eta_{(2),\mu}$$

Summarizing, we have proved the following:
*if* $\Phi$ *is given by* (28) *then*

$$< \widehat{A}(T\mathcal{F});\omega_\Phi > -\frac{1}{2} < \widetilde{\eta}_{<e>};\Phi > = < \widehat{A}(T\mathcal{F}); RS > -\frac{1}{2}\eta_{(2),\mu}$$

**Remark.** Recall that the measured eta-invariant $\eta_\mu$ of [35] is given by:

$$\sqrt{\pi}\eta_\mu = \int(\int_0^{+\infty} tr\, K(D_\partial e^{-t^2 D_\partial^2})(y,y)\, dVol_{\pi^{-1}(\theta)}(y)\, dt\,) d\mu(\theta)$$

where $D_\partial$ is the induced boundary longitudinal Dirac operator along the leaves of $(\partial M,\mathcal{F})$, $D_0$ being of course the lift of $D_\partial$. In general the two longitudinal measures

$$tr\, K(D_\partial e^{-t^2 D_\partial^2})(y,y)\, dVol_{\pi^{-1}(\theta)}(y) \neq tr\, K(D_0 e^{-t^2 D_0^2})(y,y)\, dVol_{\pi^{-1}(\theta)}(y)$$

are different. So in general $\eta_\mu \neq \eta_{(2),\mu}$. This is of course well known when $T$=point (indeed, on a Galois covering the $L^2$-eta invariant is in general not equal to the eta invariant on the base).
**End of remark.**
**End of example 2.**

**Example 3.** Let $C$ be a $\Gamma$-invariant closed current on $T$. Consider $\Phi = C\otimes\delta_e$. Then, proceeding as in the previous example, we obtain

$$\omega_\Phi = \mathcal{C}$$

with $\mathcal{C}$ the foliated closed current on $M$ induced by $C$.

Let us go back to the case of a general $\Gamma-$invariant fibration $\pi : \widehat{M}\mapsto T$. In order to prove our main result, we shall need the following transgression formula:

**Proposition 9.** *Let $u > t > 0$ and let $\widetilde{\eta}_{<e>}(s)$ be the higher eta integrand introduced in Section 5. Then*

$$\mathcal{R}({}^b STR_{<e>}\, e^{-\mathbb{A}_u^2}) \;=\; \mathcal{R}({}^b STR_{<e>}\, e^{-\mathbb{A}_t^2}) - d\int_t^u \mathcal{R}^b STR_{<e>}(\frac{d\mathbb{A}_s}{ds} e^{-\mathbb{A}_s^2})ds+ \tag{32}$$
$$-\frac{1}{2}\int_t^u \widetilde{\eta}_{<e>}(s)ds.$$

*Proof.* One proceeds exactly as in the proof of Proposition 11 of [27], using Proposition 8. □

Taking the limit as $t \to 0^+$ in the previous formula and using Theorem 2 one gets the following crucial equality in $\widehat{\overline{\Omega}}_*(T, \mathcal{B}_\Gamma^\infty)$:

$$\mathcal{R}({}^b STR_{<e>}\, e^{-\mathbb{A}_u^2}) \;=\; \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\, \mathrm{ch}(\nabla^{\widehat{V}})\, \mathrm{ch}(\nabla^{can}) - \frac{1}{2}\int_0^u \widetilde{\eta}_{<e>}(s)ds+ \tag{33}$$
$$- d\int_0^u \mathcal{R}({}^b STR_{<e>}(\frac{d\mathbb{A}_s}{ds} e^{-\mathbb{A}_s^2}))ds.$$

## 7. A higher APS-index theorem: the isometric case

### 7.1. Isometric actions.

In this subsection we shall assume that $T$ is endowed with a riemannian metric such that $\Gamma$ acts by isometries. This corresponds to a type-II situation; we shall deal with arbitrary $\Gamma$-actions in the next subsection. The following Proposition gives an important consequence of this assumption

**Proposition 10.** *Let us assume that the manifold $T$ admits a riemannian metric $g$ such that the group $\Gamma$ acts by isometries on $T$. Then $\mathcal{T}^\infty$ is closed under the holomorphic functional calculus in $C^0(T) \rtimes_r \Gamma$.*

*Proof.* $\mathcal{T}^\infty$ is equal to the set of the elements $\sum t_\gamma \gamma$ such that

$$\sup_{\theta\in T,\gamma\in\Gamma}\left[|\Delta^p t_\gamma|(\theta)(1+||\gamma||)^N\right] < \infty\ ,\ \ \forall p, N \in \mathbb{N} \tag{34}$$

where $\Delta$ is the Laplace Beltrami operator of $T$. Using the hypothesis that $\Gamma$ acts by isometries and standard arguments (see [14]) one checks that this set is indeed a subalgebra and that is closed under the holomorphic functional calculus in $C^0(T) \rtimes_r \Gamma$. Alternatively, one may check that $\mathcal{T}^\infty$ is a particular case of a construction due to Jiang (see section 3 of [11] ). □

In particular

$$K_0(\mathcal{T}^\infty) \simeq K_0(C^0(T) \rtimes \Gamma). \tag{35}$$

Using the fact that $\mathcal{T}^\infty$ is dense and stable under holomorphic functional calculus and proceeding as in the proof of the decomposition theorem of [17] (see Theorem 12.7 there) we can give a very explicit decription of the index class in $K_0(\mathcal{T}^\infty)$. Briefly, we can find

$\epsilon > 0$, and $\mathcal{L}_\infty$ [resp. $\mathcal{N}_\infty$] a sub-$\mathcal{T}^\infty$−module projective of finite rank of $x^\epsilon H^\infty_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E}^+)$ [resp. $x^\epsilon H^\infty_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E}^-)$] such that

$$\operatorname{Ind}(D^+) = [\mathcal{L}_\infty] - [\mathcal{N}_\infty] \text{ in } K_0(\mathcal{T}^\infty) \tag{36}$$

Further properties of these sub-modules can be found in Appendix C.

7.2. **The APS index theorem for the groupoid $T \rtimes \Gamma$ in the isometric case.**

Recall that $\mathrm{Ch}(\operatorname{Ind} D^+)$ is defined (thanks to the work of [12]) by a representative in $\widehat{\Omega}_*(\mathcal{T}^\infty)$ whereas the $b$−supertrace of the superconnection heat kernel is defined by a representative in $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)$. In order to establish a connection between these two objects we state the following proposition whose easy proof will be left to the reader.

**Proposition 11.**
*1). By universality there is a natural morphism of algebras:*

$$j : \widehat{\Omega}_*(\mathcal{T}^\infty) \to \widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma).$$

*2). Let $\mathcal{F} = \mathcal{L} \oplus \mathcal{N}$ be a $\mathbb{Z}_2$−graded projective finitely generated $\mathcal{T}^\infty$−module. Let $\nabla$ be a $\mathcal{T}^\infty$−connection of $\mathcal{F}$ with value in $\widehat{\Omega}_1(\mathcal{T}^\infty) \otimes \mathcal{F}$ and preserving both $\mathcal{L}$ and $\mathcal{N}$. Then $\widetilde{\nabla} = (j \otimes \mathrm{Id}_\mathcal{F}) \circ \nabla$ defines a connection of $\mathcal{F}$ with values in $\widehat{\Omega}_1(T, \mathcal{B}^\infty_\Gamma) \otimes \mathcal{F}$.*
*3). The two supertraces (in the algebraic sense) $\mathcal{R}STR^{alg}_{<e>} e^{-\widetilde{\nabla}^2}$ and $j_{<e>}(\mathcal{R}STR^{alg} e^{-\nabla^2})$ coincide in $\overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma)$ (ie modulo graded commutators). Moreover for any $\mathcal{T}^\infty$−connection $\nabla_1$ of $\mathcal{F}$ with values in $\widehat{\Omega}_1(\mathcal{T}^\infty) \otimes \mathcal{F}$ and preserving both $\mathcal{L}$ and $\mathcal{N}$, the two supertraces $\mathcal{R}STR^{alg}_{<e>} e^{-\widetilde{\nabla}^2}$ and $\mathcal{R}STR^{alg}_{<e>} e^{-{\nabla_1}^2}$ coincide modulo an exact form.*

Now we may state the higher index theorem for the groupoid $T \rtimes \Gamma$ and the proper$T \rtimes \Gamma$−manifold $\widehat{M}$.

**Theorem 3.** *Let $\Gamma$ be of polynomial growth and let $D^+ = (D^+(\theta))_{\theta \in T}$ a $\Gamma$-equivariant family of Dirac operators with inverible boundary family $D_0$. Then the following formula holds in the homology $\widehat{H}_*(T, \mathcal{B}^\infty_\Gamma)$ (see Section 5.1):*

$$j_{<e>}(\mathrm{Ch}\ \operatorname{Ind} D^+) = \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) - \frac{1}{2}\widetilde{\eta}_{<e>} \in \widehat{H}_*(T, \mathcal{B}^\infty_\Gamma)$$

*$j_{<e>}(\mathrm{Ch}\ \operatorname{Ind} D^+)$ denotes the projection of $j(\mathrm{Ch}\ \operatorname{Ind} D^+)$ on the set of differential forms $\sum_{\gamma_0\gamma_1\cdots\gamma_k = e} \omega_{\gamma_0,\cdots,\gamma_k} \otimes \gamma_0 d\gamma_1 \cdots d\gamma_k$ of $\widehat{\Omega}_\star(T, \mathcal{B}^\infty_\Gamma)$ which are concentrated (with respect to the group variables) in the trivial conjugacy class $< e >$ of $\Gamma$.*

**Definition 10.** *Let $\Phi$ be a closed graded trace on $\widehat{\Omega}_\star(T, \mathcal{B}^\infty_\Gamma)$ as above. We define the higher $\Phi$-index of $D^+$ as*

$$\operatorname{Ind}_\Phi(D^+) := < j_{<e>}(\mathrm{Ch}\ \operatorname{Ind} D^+), \Phi > \ .$$

**Corollary 2.** *For the higher $\Phi$-index of a $\Gamma$-equivariant family as above the following formula holds:*

$$\operatorname{Ind}_\Phi(D^+) =< \widehat{A}(T\mathcal{F})\,\mathrm{ch}(\nabla^V)\,,\, \omega_\Phi > - < \frac{1}{2}\widetilde{\eta}_{<e>}\,;\, \Phi >$$

The corollary follows from the Theorem, using the discussion in Subsection 6.2. The proof of the theorem can be found in Appendix C.

## 8. A higher APS index theorem: the general case

### 8.1. The Chern character of the index class.

In the general case, when $\Gamma$ does not preserve any Riemannian metric on $T$, it is not known whether the algebra $\mathcal{T}^\infty$ is stable (or not stable) under the holomorphic functional calculus in $C^0(T) \rtimes_r \Gamma$. For this reason, the definition of the Chern character of the index class and of its pairing with a closed graded trace on $\widehat{\Omega}_*(T, \mathcal{B}_\Gamma^\infty)$ is more involved. We shall devote the beginning of this subsection to a few preliminaries (following [7] and [9]) leading to such a definition. We follow the notations in [9].

We consider the following algebras:

$$\mathcal{U} = \mathcal{T}^\infty = C^\infty(T, \mathcal{B}_\Gamma^\infty)\,, \quad \Omega^* = \widehat{\Omega}_*(T, \mathcal{B}_\Gamma^\infty)$$
$$\widetilde{\mathcal{U}} = \Psi_{\mathcal{T}^\infty}^{-\infty,\delta}(\widehat{M}, \widehat{E}^+)\,, \quad \widetilde{\Omega}^* = \Psi_{\widehat{\Omega}_*(T,\mathcal{B}_\Gamma^\infty)}^{-\infty,\delta}(\widehat{M}, \widehat{E}^+).$$

Using a $\Gamma-$equivariant complex vector bundle isomorphism: $\widehat{E}^+ \simeq \widehat{E}^-$ we get natural identifications:

$$\Psi_{\mathcal{T}^\infty}^{-\infty,\delta}(\widehat{M}, \widehat{E}^+) \simeq \Psi_{\mathcal{T}^\infty}^{-\infty,\delta}(\widehat{M}, \widehat{E}^-), \quad \Psi_{\widehat{\Omega}_*(T,\mathcal{B}_\Gamma^\infty)}^{-\infty,\delta}(\widehat{M}, \widehat{E}^+) \simeq \Psi_{\widehat{\Omega}_*(T,\mathcal{B}_\Gamma^\infty)}^{-\infty,\delta}(\widehat{M}, \widehat{E}^-)$$

that we shall use in this subection. Then we consider the $\mathcal{T}^\infty-$module $\mathcal{E} = H^\infty_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$ and (with the notations defining $\mathbb{A}_s$) the connection $\nabla = \nabla^u + \sum_{\gamma\in\Gamma} d\gamma \otimes h\gamma^{-1}$.Observe that $\nabla$ sends $\mathcal{E}$ into $\Omega_1(T, \mathcal{B}^\infty) \otimes_{\mathcal{T}^\infty} \mathcal{E}$ and that one defines a degree one map $\widetilde{\nabla} : \widetilde{\Omega}^* \mapsto \widetilde{\Omega}^{*+1}$ by setting, for any $K \in \widetilde{\Omega}^*$, $\widetilde{\nabla}(K) = [\nabla, K] \in \widetilde{\Omega}^{*+1}$, where $[\nabla, K]$ denotes the graded commutator. Now set $\Theta = \nabla^2$, $\Theta$ sends $\mathcal{E}$ into $\Omega_2(T, \mathcal{B}^\infty) \otimes_{\mathcal{T}^\infty} \mathcal{E}$ but is not given by a smooth integral kernel in $\widetilde{\Omega}^*$, nevertheless for any $K \in \widetilde{\Omega}^*$, $\Theta K$ and $K\Theta$ both belong to $\widetilde{\Omega}^*$ One then checks that for any $K \in \widetilde{\Omega}^*$, $\widetilde{\nabla}^2(K) = \Theta K - K\Theta$.

Let $\Phi$ a closed graded trace on $\widehat{\Omega}_*(T, \mathcal{B}_\Gamma^\infty)$, concentrated on the trivial conjugacy class. Then $\Phi$ extends to a closed graded trace $\widetilde{\Phi}$ on $\widetilde{\Omega}^*$:

$$\widetilde{\Phi}(K) := \Phi(\mathcal{R}\,\mathrm{TR}_{<e>}(K)).$$

The forthcoming Lemma 14 states that for any $K \in \widetilde{\Omega}^*$,

$$\mathcal{R}\,\mathrm{STR}_{<e>}\,[\nabla, K] = d\,\mathcal{R}\,\mathrm{STR}_{<e>}\,K \quad \text{modulo graded commutators.}$$

Thus, for any $K \in \widetilde{\Omega}^*$, $\widetilde{\Phi}(\widetilde{\nabla}(K)) = 0$. Next we recall Connes' construction ([7] page 229) of a cycle $(\widetilde{\widetilde{\Omega}}^*, d, \widetilde{\widetilde{\Phi}})$ over $\widetilde{\mathcal{U}} = \widetilde{\Omega}^0$. Let $X$ be a new odd formal variable of degree 1 and put:

$$\widetilde{\widetilde{\Omega}}^* = \widetilde{\Omega}^* \oplus X\widetilde{\Omega}^* \oplus \widetilde{\Omega}X^* \oplus X\widetilde{\Omega}^*X$$

with the new multiplication rules $(\widetilde{\omega}_1 X)\widetilde{\omega}_2 = \widetilde{\omega}_1(X\widetilde{\omega}_2) = 0$ and $(\widetilde{\omega}_1 X)(X\widetilde{\omega}_2) = \widetilde{\omega}_1\Theta\widetilde{\omega}_2$. Then define a new graded trace on $\widetilde{\widetilde{\Omega}}^*$ by the formula:

$$\widetilde{\widetilde{\Phi}}(\widetilde{\omega}_1 + X\widetilde{\omega}_2 + \widetilde{\omega}_3 X + X\widetilde{\omega}_4 X) = \widetilde{\Phi}(\widetilde{\omega}_1) + (-1)^{\partial\widetilde{\omega}_4}\widetilde{\Phi}(\widetilde{\omega}_4\Theta).$$

Define a differential $d$ on $\widetilde{\widetilde{\Omega}}^*$ which is generated by the relations

$$d\widetilde{\omega} = \widetilde{\nabla}(\widetilde{\omega}) + X\widetilde{\omega} + (-1)^{\partial\widetilde{\omega}}\widetilde{\omega}X$$

and $dX = 0$. One can check that $d^2 = 0$ and that $\tilde{\tilde{\Phi}}(d\tilde{\tilde{\omega}}) = 0$ for any $\tilde{\tilde{\omega}} \in \tilde{\tilde{\Omega}}^*$. Therefore, $(\tilde{\tilde{\Omega}}^*, d, \tilde{\tilde{\Phi}})$ defines a cycle over $\widetilde{\mathcal{U}} = \widetilde{\Omega}^0$ of dimension equal to the degree of $\widetilde{\Phi}$. Let $\widehat{\tau}_\Phi$ be cyclic cocycle of $\widetilde{\mathcal{U}}$ defined by the character (see [7] page 186) of this cycle. Let $\widetilde{\mathcal{U}}^+$ be the algebra obtained by adding a unit to $\widetilde{\mathcal{U}}$ with canonical homomorphism $\widehat{\pi} : \widetilde{\mathcal{U}}^+ \to \mathbb{C}$. Let us recall how $\widehat{\tau}_\Phi$ induces a cyclic cocycle, still denoted $\widehat{\tau}_\Phi$, on $\widetilde{\mathcal{U}}^+$. Set $\tilde{\tilde{\Omega}}^0_+ = \widetilde{\widetilde{\Omega}^0} \oplus \mathbb{C} = \widetilde{\mathcal{U}}^+$ and $\tilde{\tilde{\Omega}}^k_+ = \tilde{\tilde{\Omega}}^k$ for any $k \in \mathbb{N}^*$. Then define $d : \tilde{\tilde{\Omega}}^0_+ \to \tilde{\tilde{\Omega}}^1$ by the formula $d(\omega_0 \oplus \lambda) = d\omega_0$ (ie, $d\lambda = 0$ for any $\lambda \in \mathbb{C}$), the differential from $\tilde{\tilde{\Omega}}^k_+$ to $\tilde{\tilde{\Omega}}^{k+1}_+$ remaining unchanged for $k \geq 1$. If the degree of $\widetilde{\Phi}$ is zero then $\widehat{\tau}_\Phi$ is a trace on $\widetilde{\Omega}^0$ which extends as a trace on $\tilde{\tilde{\Omega}}^0_+$ by the formula $\widehat{\tau}_\Phi(\omega_0 \oplus \lambda) = \widehat{\tau}_\Phi(\omega_0)$. If the degree of $\widetilde{\Phi}$ is equal to $n \geq 1$ then one gets a cycle $(\tilde{\tilde{\Omega}}^*_+, d, \tilde{\tilde{\Phi}})$ over $\widetilde{\mathcal{U}}_+$ which allows to define a cyclic cocycle still denoted $\widehat{\tau}_\Phi$ on $\widetilde{\mathcal{U}}^+$ by the formula:

$$\widehat{\tau}_\Phi(\omega_0 \oplus \lambda_0, \cdots, \omega_n \oplus \lambda_n) = \widehat{\tau}_\Phi(\omega_0, \cdots, \omega_n).$$

Consider $p$ a projection in $M_2(\widetilde{\mathcal{U}}^+)$ such that $\widehat{\pi}(p) = \widehat{\pi}(p_0)$ where

$$p_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then Connes has defined the pairing $< [p - p_0], [\widehat{\tau}_\Phi] >$ ([7] page 224) between the $K-$theory class $[p - p_0]$ and the cyclic cocycle $\widehat{\tau}_\Phi$. We shall use the following lemma.

**Lemma 9.** *The operator $pe^{(p\nabla p)^2}p - p_0 e^{(p_0 \nabla p_0)^2} p_0$ belongs to $M_2(\widetilde{\Omega}^*)$ and there exists a constant $C(\deg \Phi)$ depending only on the degree of $\Phi$ such that:*

$$C(\deg \Phi) < [p - p_0], [\widehat{\tau}_\Phi] > = < \mathcal{R}\, \mathrm{TR}_{<e>} \left( pe^{(p\nabla p)^2} p - p_0 e^{(p_0 \nabla p_0)^2} p_0 \right), \Phi >$$

*where the definition of $\mathrm{TR}_{<e>}$ is given as in Definition 8 simply replacing there $Str$ by $tr \otimes Tr_{M_2(\mathbb{C})}$.*

*Proof.* Let $(e_{i,j})_{1 \leq i,j \leq 2}$ be the basis formed with elementary matrix of $M_2(\mathbb{C})$. Write $p = \sum_{i,j} p_{i,j} e_{i,j}$. Then recall that:

$$\widehat{\tau}_\Phi \sharp Tr(p, \cdots, p) = \sum_{i_0, j_0, \cdots, i_n, j_n} \widehat{\tau}_\Phi(p_{i_0,j_0}, \cdots, p_{i_n,j_n})\, Tr(\, e_{i_0,j_0} \cdots e_{i_n,j_n}\, ).$$

Next using the fact (see [9] just before (5.5)) that $p\, dpdp\, p$ equals the curvature of $p \circ \nabla \circ p$ one proves easily the Lemma.

□

Now let $P$ be a parametrix of $D^+$ then there exists $\delta > 0$ such that $S_+ = \mathrm{Id} - PD^+$ and $S_- = \mathrm{Id} - D^+ P$ both belong to $\Psi^{-\infty,\delta}_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E}^+) \simeq \Psi^{-\infty,\delta}_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E}^-)$ and, in particular, $I(S_\pm; \lambda) \equiv 0$. As in [9] (section 5) and [7] we consider the two projections $p$ and $p_0$ defined by:

$$p = \begin{pmatrix} S_+^2 & S_+(\mathrm{Id} + S_+)P \\ S_- D^+ & \mathrm{Id} - S_-^2 \end{pmatrix}, \quad p_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall (see [7] page 131) that the $K-$theory class $\operatorname{Ind} D^+ \in K_0(\Psi^{-\infty,\delta}_{\mathcal{I}^\infty}(\widehat{M},\widehat{E}^+))$ is defined to be equal to $[p - p_0]$.

Being motivated by section 5 of [9] we give the following:

**Definition 11.** *We define* $\mathrm{Ch}_{<e>} \operatorname{Ind} D^+$ *to be equal to*

$$\mathrm{Ch}_{<e>} \operatorname{Ind} D^+ = \mathcal{R}\,\mathrm{TR}_{<e>}\left(pe^{(p\nabla p)^2}p - p_0 e^{(p_0\nabla p_0)^2}p_0\right).$$

**Definition 12.** *Let* $\Phi$ *a closed graded trace on* $\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)$*, concentrated on the trivial conjugacy class. Let* $\widetilde{\Phi}$ *be its extension to* $\Psi^{-\infty,\delta}_{\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}$*. We define the higher* $\Phi$*-index of* $D^+$ *as*

$$\operatorname{Ind}_\Phi(D^+) := <\mathrm{Ch}_{<e>}(\operatorname{Ind} D^+)\,;\,\widetilde{\Phi}>$$

Lemma 9 shows that this definition is compatible with the one of Connes.

8.2. **The Chern character of a superconnection.**

We consider again

$$\nabla = \nabla^u + \sum_{\gamma\in\Gamma} d\gamma \otimes h\gamma^{-1}$$

and we define a new connection $\nabla' = \nabla'^{,+} \oplus \nabla'^{,-}$ acting on $C^\infty_{\mathcal{I}^\infty}(\widehat{M},\widehat{E})$ by:

$$\nabla'^{,+} = S_+\nabla^+ S_+ + P(\mathrm{Id} + S_-)\nabla^- D^+,\ \nabla'^{,-} = \nabla^-.$$

As in section 5 of [9] we consider the superconnection $A(t)$ defined for $t \geq 0$ by

$$A(t) = \begin{pmatrix} \nabla'^{,+} & tD^- \\ tD^+ & \nabla'^{,-} \end{pmatrix}$$

On the one hand this connection is closely related to the operators appearing in the definition of the index class; on the other hand, precisely for this reason, the boundary behaviour of $\nabla'$ is rather complicated and it is a source of technical complications in the proof of our main theorem.

**Definition 13.** *For each* $t \geq 0$ *we define the Chern character of* $A(t)$*, with values in* $\overline{\widehat{\Omega}}_*(T,\mathcal{B}^\infty_\Gamma)$*, as follows*

$$\mathrm{Ch}_{<e>}\ A(t) = \mathcal{R}\left(TR_{<e>}S_+ e^{-A^2(t)}_{1,1} S_+ + TR_{<e>}(De^{-A^2(t)}_{1,1} P(\mathrm{Id} + S_-) - e^{-A^2(t)}_{2,2})\right). \tag{37}$$

Notice that the first summand on the right hand side is an element in $\Psi^{-\infty,\delta}_{\widehat{\Omega}(T,\mathcal{B}^\infty_\Gamma)}$, thus the trace is well defined. A simple computation shows that the operator $(De^{-A^2(t)}_{1,1} P(\mathrm{Id} + S_-) - e^{-A^2(t)}_{2,2})$, which a priori belongs to $\Psi^{-\infty,\delta}_{b,\widehat{\Omega}(T,\mathcal{B}^\infty_\Gamma)}$, has vanishing indicial family; thus the trace of the second summand on the right hand side also exists.

We shall see below, in Proposition 12, that $\mathrm{Ch}_{<e>}\ A(t)$ is closed for $t \geq 0$.

**Lemma 10.** *The following formula holds:*

$$\mathrm{Ch}_{<e>}\ A(0) = \mathrm{Ch}_{<e>}\ \operatorname{Ind} D^+. \tag{38}$$

*Proof.* This follows from Definition 11 and formula (5.29) in [9]. □

It is important to observe that for $t > 0$ the noncommutative differential form $\mathcal{R}\,{}^bSTR_{<e>}e^{-A^2(t)}$ is well defined.

8.3. **Main theorem and strategy of the proof.**

The main result of this paper is the following higher APS-index theorem

**Theorem 4.**
*1) The following formula holds in the homology $\widehat{H}_*(T,\mathcal{B}_\Gamma^\infty)$ (see Subsection 5.1):*

$$\mathrm{Ch}_{<e>}(\mathrm{Ind}\, D^+) \;=\; \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) - \frac{1}{2}\widetilde{\eta}_{<e>} \in \widehat{H}_*(T,\mathcal{B}_\Gamma^\infty)$$

*2) Let $\Phi$ a closed graded trace on $\widehat{\Omega}_*(T,\mathcal{B}_\Gamma^\infty)$, concentrated on the trivial conjugacy class. Then there exists a current $\omega_\Phi$ on $M$ such that the following formula holds:*

$$\mathrm{Ind}_\Phi(D^+) = < \widehat{A}(T\mathcal{F})\,\mathrm{ch}(\nabla^V)\,,\;\omega_\Phi > - < \frac{1}{2}\widetilde{\eta}_{<e>}\,;\,\Phi >$$

**Strategy of the proof of Theorem 4 .**
The proof of 1) can be divided into 4 steps.

**Step 1.** For any real $t>0$ one has

$$\mathcal{R}^b STR_{<e>}e^{-\mathbb{A}_t^2} = \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) - \frac{1}{\sqrt{\pi}}\int_0^t \widetilde{\eta}_{<e>}(s)ds.$$

modulo an exact form.

**Step 2.** For any real $t>0$ one has

$$\mathcal{R}^{\,b}STR_{<e>}e^{-A^2(t)} = \mathcal{R}^{\,b}STR_{<e>}e^{-\mathbb{A}_t^2} + B_1(t)$$

modulo an exact form, with $B_1(t)$ a boundary term which goes to zero as $t\to+\infty$ in $\overline{\widehat{\Omega}}_*(T,\mathcal{B}_\Gamma^\infty)$ endowed with the Hausdorff quotient topology induced by the Frechet topology of $\widehat{\Omega}_*(T,\mathcal{B}_\Gamma^\infty)$.

**Step 3.** For $t>0$ one has

$$\mathrm{Ch}_{<e>}\;A(t)\;-\;\mathcal{R}^{\,b}STR_{<e>}e^{-A^2(t)} = B_2(t)$$

with $B_2(t)$ a boundary term going to zero as $t\to+\infty$.

**Step 4.** For any real $t>0$,

$$\mathrm{Ch}_{<e>}\;A(t) = \mathrm{Ch}_{<e>}\;A(0)\,(=\mathrm{Ch}_{<e>}\,\mathrm{Ind}\,D^+)$$

modulo an exact form.

**Final step.**
First of all, one deduces 2) from 1) exactly as explained in the proof of Corollary 1.
In order to prove 1) we proceed as follows: by Step 4, Step 3, Step 2, Step 1 we have for each $t>0$, modulo exact forms,

$$\begin{aligned}
\mathrm{Ch}_{<e>}\,\mathrm{Ind}\,D^+ \;&=\; \mathrm{Ch}_{<e>}\,A(t)\\
&=\; \mathcal{R}^{\,b}STR_{<e>}e^{-A^2(t)} + B_2(t)\\
&=\; \mathcal{R}^{\,b}STR_{<e>}e^{-\mathbb{A}_t^2} + B_1(t) + B_2(t)\\
&=\; \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) - \frac{1}{\sqrt{\pi}}\int_0^t \widetilde{\eta}_{<e>}(s)ds + B_1(t) + B_2(t)
\end{aligned}$$

with $B_1(t)$ and $B_2(t)$ additional boundary terms that go to zero as $t \to +\infty$. The index formula thus follows by taking $t \to +\infty$. This finishes the sketch of the proof of the main theorem.

We shall not use the following result but it is interesting (and reassuring) in itself.

**Proposition 12.** *For each real $t \geq 0$, $\mathrm{Ch}_{<e>} A(t)$ is closed.*

*Proof.* (Sketch) Lemma 14 implies that

$$d\mathcal{R}\,{}^b STR_{<e>} e^{-\mathbb{A}_t^2} = \mathcal{R}\,{}^b STR_{<e>} [\nabla,\, e^{-\mathbb{A}_t^2}] = \mathcal{R}\,{}^b STR_{<e>} [\nabla - \mathbb{A}_t,\, e^{-\mathbb{A}_t^2}].$$

Using Proposition 8 and the fact that $D_0$ is invertible, one checks (after an easy computation) that

$$\lim_{t\to+\infty} \mathcal{R}\,{}^b STR_{<e>} [\nabla - \mathbb{A}_t,\, e^{-\mathbb{A}_t^2}] = 0.$$

So $\lim_{t\to+\infty} d\mathcal{R}\,{}^b STR_{<e>} e^{-\mathbb{A}_t^2} = 0$ and, Steps 2 and 3 imply that $\lim_{t\to+\infty} d\,\mathrm{Ch}_{<e>} A(t) = 0$. The Lemma is then a consequence of Step 4. $\square$

## 9. **Proof of the main theorem**

We shall first prove Step 3. We shall then give a detailed proof of Step 4; this is the most intricate step and will require 4 lemmas. Then we shall only sketch the proof of Step 2, since it is similar to Step 4 but simpler. Notice that Step 1 has already been proved.

### 9.1. **Proof of Step 3.**

We consider the noncommutative differential form

$$\mathcal{R}\,{}^b TR_{<e>} [D^+, e_{1,1}^{-A^2(t)} P] + \mathcal{R}\,{}^b STR_{<e>} e^{-A^2(t)}.$$

(Notice that it would be more rigorous to write

$$\mathcal{R}\,{}^b TR_{<e>} D^+ e_{1,1}^{-A^2(t)} P - \mathcal{R}\,{}^b TR_{<e>} e_{1,1}^{-A^2(t)} P D^+$$

instead of $\mathcal{R}\,{}^b TR_{<e>} [D^+, e_{1,1}^{-A^2(t)} P]$ as the operators involved are not endomorphisms but rather homomorphisms from the section of $\widehat{E}^{\pm}$ to the sections of $\widehat{E}^{\mp}$. )

**Lemma 11.** *For each real $t > 0$ one has:*

$$\mathrm{Ch}_{<e>} A(t) = \mathcal{R}\,{}^b TR_{<e>} [D^+, e_{1,1}^{-A^2(t)} P] + \mathcal{R}\,{}^b STR_{<e>} e^{-A^2(t)}.$$

*Proof.* It is a simple computation using the definition of $S_{\pm}$ and the fact that the indicial family of $(De_{1,1}^{-A^2(t)} P(\mathrm{Id} + S_-) - e_{2,2}^{-A^2(t)})$ is identically zero. The easy details are left to the reader. $\square$

Using Proposition 8, an easy computation shows that:

**Lemma 12.**

$$\mathcal{R}\,{}^b TR_{<e>} [D^+, e_{1,1}^{-A^2(t)} P] = -\frac{1}{2\pi} \mathcal{R} TR\big( e^{-(\nabla_\partial^-)^2 - t^2 D_0^2} \int_{\mathbb{R}} e^{-t^2\lambda^2} (D_0 + \sqrt{-1}\lambda\,\mathrm{Id})^{-1}\, d\lambda \big). \tag{39}$$

**Lemma 13.** *As $t \to +\infty$*

$$\mathcal{R}TR\big( e^{-(\nabla_\partial^-)^2 - t^2 D_0^2} \int_{\mathbb{R}} e^{-t^2\lambda^2} (D_0 + \sqrt{-1}\lambda \,\mathrm{Id})^{-1} \, d\lambda \big) \longrightarrow 0 \tag{40}$$

*in the space of noncommutative differential forms $\overline{\widehat{\Omega}}_*(T, \mathcal{B}_\Gamma^\infty)$.*

*Proof.* Since $D_0$ is invertible, the following finite propagation speed estimates (see [23] page 215) are valid. There exists $\delta > 0$ such that:

$$\forall s \geq 1, \ \forall (y,z) \in \partial\widehat{M} \times_\pi \partial\widehat{M}, \ \ |e^{-s^2 D_0^2}|(y,z) \leq C(N)(1 + d(y,z))^{-N} e^{-s^2\delta}. \tag{41}$$

Now, using Duhamel's formula, we may write:

$$e^{-(\nabla_\partial^-)^2 - t^2 D_0^2} = e^{-t^2 D_0^2} + \int_0^1 e^{-ut^2 D_0^2} (-(\nabla_\partial^-)^2) e^{-(1-u)t^2 D_0^2} \, du + \cdots +$$

$$\int_{\Delta_k} e^{-u_0 t^2 D_0^2} (-(\nabla_\partial^-)^2) e^{-u_1 t^2 D_0^2} (-(\nabla_\partial^-)^2) \cdots e^{-u_k t^2 D_0^2} d\sigma(u_0, \cdots, u_k) + \ldots$$

where $\Delta_k$ denotes the $k-$simplex:

$$\Delta_k = \{(u_0, \cdots, u_k) \in [0,1]^{k+1} / , \ \sum_{j=0}^k u_j = 1\}.$$

It is clear that for any $(u_0, \cdots, u_k) \in \Delta_k$, at least one of the $u_j$ satisfies $u_j \geq \frac{1}{k+1}$. Observe moreover that standard finite propagation speed estimates allow to show that the Schwartz kernel of $(D_0 + \sqrt{-1}\lambda \,\mathrm{Id})^{-1}$ is rapidly decreasing outside the diagonal. Using the estimates (41), one checks easily that

$$K_k(t) = \int_{\Delta_k} e^{-u_0 t^2 D_0^2} (-(\nabla_\partial^-)^2) e^{-u_1 t^2 D_0^2} (-(\nabla_\partial^-)^2) \cdots e^{-u_k t^2 D_0^2} d\sigma(u_0, \cdots, u_k) + \ldots$$

belongs to the space of Definition 7 and that the supremum constants (for $K_k(t)$) of the last line of Definition 7 are lower than $Ce^{-\frac{\delta t^2}{2k+2}}$. For a bit more details the reader may read the proof of Theorem 2.9 of [17]. One then gets immediately the Lemma.

□

Step 3 follows from the above three lemmas.

## 9.2. **Proof of Step 4.**

The following lemma appears implicitly in the literature, we feel it is appropriate to give a detailed proof. We thank Sasha Gorokhovsky for useful explanations.

**Lemma 14.** *Let $K$ be an element of $\Psi^{-\infty}_{b, \widehat{\Omega}_*(T, \mathcal{B}_\Gamma^\infty)}(\widehat{M}; \widehat{E})$. Then*

$${}^b STR_{<e>} [\nabla, K] = d \, {}^b STR_{<e>} K$$

*modulo graded commutators where we recall that $\nabla = \nabla^u + \sum_{\gamma \in \Gamma} d\gamma \otimes h\gamma^{-1}$.*

*Proof.* Recall that the differential $d$ of $\overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma)$ may be written as $d = d_T + d_\Gamma$ where $d_T$ [resp. $d_\Gamma$] is the differential corresponding to $\Omega^*(T)$ [resp. $\Omega^*(\mathcal{B}^\infty_\Gamma)$]. The fact that ${}^bSTR_{<e>}[\nabla^u, K] = d_T\, {}^bSTR_{<e>} K$ is basically proved in [27]. So we have to prove that ${}^bSTR_{<e>}[A, K] = d_\Gamma\, {}^bSTR_{<e>} K$ where we have set $A = \sum_{\gamma\in\Gamma} d\gamma \otimes h\gamma^{-1}$.

To shorten the notations we write $K(z,w) = \sum_\omega \omega K_\omega(z,w)$ where it is understood (as in section 6.1) that $\omega = dg_1 \cdots dg_k$ where $g_1, \dots, g_k \in \Gamma$. We set $\pi_1(\omega) = g_1 \cdots g_k$. Since $K$ is (graded-)$\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma)-$linear, we observe that $KA$ can be written as $(-1)^{|\omega|}\sum_{\gamma,\omega} d\gamma\gamma^{-1}\omega K_\omega h^\gamma$ where $K_\omega h^\gamma$ denotes the kernel $K_\omega(z,w)h^\gamma(w)$ and $h^\gamma(w) = h(w\cdot\gamma)$ . Now using the definition of the action of the connection $A$ on acting on $\widehat{\Omega}_*(T, \mathcal{B}^\infty_\Gamma) \otimes_{\mathcal{T}^\infty} C^\infty_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$ , one sees that $AK$ can be written as $\sum_{\gamma,\omega}(-1)^{|\omega|}\omega d\gamma\gamma^{-1}h^\gamma K_\omega$, where $h^\gamma K_\omega$ denotes the kernel $h^\gamma(z)K_\omega(z,w)$.

Let $H$ be an operator (not necessarily $\mathcal{T}^\infty-$linear, such as $AK$ or $KA$) given by a Schwartz kernel $H(z,w)$ of the form $H(z,w) = \sum_{\widetilde{\omega}} \widetilde{\omega} H_{\widetilde{\omega}}(z,w)$ where it is understood that $\widetilde{\omega} = g_0 dg_1 \cdots dg_k$ and such that the $H_{\widetilde{\omega}}(z,w)$ satisfy the decay estimates of Definition 20. Then we set for any $\theta \in T$:

$$TR_{<e>}H(\theta) = \sum_{\widetilde{\omega}} \widetilde{\omega}\pi_1(\widetilde{\omega})^{-1}\, {}^\nu\!\!\int H_{\widetilde{\omega}}(w\pi_1(\widetilde{\omega}), w)\phi(w) dVol^b_{\pi^{-1}(\theta)}(w)$$

where $\pi_1(\widetilde{\omega}) = g_0 g_1 \cdots g_k$.

Notice that in the formula expressing $KA$ we have $d\gamma\gamma^{-1} = -\gamma^{-1}d\gamma$. Hence for any $\theta \in T$ we get:

$$\begin{aligned} TR_{<e>}[K, A](\theta) \;&=\; \sum_{\gamma,\omega} (-1)^{|\omega|} d\gamma\gamma^{-1}\omega\pi_1(\omega)^{-1}\, {}^\nu\!\!\int \phi(w)K_\omega(w\pi_1(\omega), w)h^\gamma(w) dVol^b_{\pi^{-1}(\theta)}(w) \\ &\quad - \sum_{\gamma,\omega} \omega d\gamma\gamma^{-1}\pi_1(\omega)^{-1}\, {}^\nu\!\!\int \phi(w)h^\gamma(w\pi_1(\omega))K_\omega(w\pi_1(\omega), w) dVol^b_{\pi^{-1}(\theta)}(w) \end{aligned}$$

and modulo graded commutators this equals to

$$\sum_{\gamma,\omega} \omega\pi_1(\omega)^{-1} d\gamma\gamma^{-1}\, {}^\nu\!\!\int \phi(w)K_\omega(w\pi_1(\omega), w)h^\gamma(w) dVol^b_{\pi^{-1}(\theta)} -$$

$$\sum_{\gamma,\omega} \omega d\gamma\gamma^{-1}\pi_1(\omega)^{-1}\, {}^\nu\!\!\int \phi(w)h^\gamma(w\pi_1(\omega))K_\omega(w\pi_1(\omega), w) dVol^b_{\pi^{-1}(\theta)}$$

The second term, after the renaming $\gamma' = \pi_1(\omega)\gamma$ (then replace $\gamma'$ by $\gamma$) can be written as

$$\sum_{\gamma,\omega} \omega d(\pi_1(\omega)^{-1}\gamma)\gamma^{-1}\, {}^\nu\!\!\int \phi(w)h^\gamma(w)K_\omega(w\pi_1(\omega), w) dVol^b_{\pi^{-1}(\theta)} =$$

$$\sum_{\gamma,\omega} \omega d(\pi_1(\omega)^{-1})\, {}^\nu\!\!\int \phi(w)h^\gamma(w)K_\omega(w\pi_1(\omega), w) dVol^b_{\pi^{-1}(\theta)} +$$

$$\sum_{\gamma,\omega} \omega\pi_1(\omega)^{-1} d\gamma\gamma^{-1}\, {}^\nu\!\!\int \phi(w)h^\gamma(w)K_\omega(w\pi_1(\omega), w) dVol^b_{\pi^{-1}(\theta)}$$

Hence $TR_{<e>}[K,A](\theta) = -\sum_{\gamma,\omega} \omega d(\pi_1(\omega)^{-1}) {}^\nu\!\!\int \phi(w)h^\gamma(w)K_\omega(w\pi_1(\omega),w)dVol^b_{\pi^{-1}(\theta)} = (-1)^{|\omega|+1}dTR_{<e>}(K)$, as $\sum_{\gamma\in\Gamma} h^\gamma = 1$ and $d(\omega\pi_1(\omega)^{-1}) = (-1)^{|\omega|}\omega d\pi_1(\omega)^{-1}$. □

Now we recall from formulas (5.25) and (5.26) of [9] that modulo operators with vanishing indicial families we have the following two identities:

$$A^2(t) \equiv \begin{pmatrix} (\nabla'^{,+})^2 + t^2D^-D^+ & t[\nabla'^{,-},D^-] + t(\nabla'^{,+} - \nabla'^{,-})D^- \\ 0 & D^+(\,(\nabla'^{,+})^2 + t^2D^-D^+\,)P \end{pmatrix} \tag{42}$$

$$e^{-A^2(t)} \equiv \begin{pmatrix} e^{-(\nabla'^{,+})^2 - t^2D^-D^+} & \mathcal{Z} \\ 0 & De^{-(\nabla'^{,+})^2 - t^2D^-D^+}P \end{pmatrix} \tag{43}$$

with $\mathcal{Z}$ given by

$$-\int_0^1 e^{-u(\,(\nabla'^{,+})^2 + t^2D^-D^+\,)}\left(t[\nabla'^{,-},D^-] + t(\nabla'^{,+} - \nabla'^{,-})D^+\right)e^{-(1-u)(\,(\nabla'^{,+})^2 + t^2D^+D^-\,)}du$$

**Lemma 15.** *The following formula holds:*

$$\begin{aligned}\frac{d}{dt}\mathcal{R}^{\,b}STR_{<e>}\,e^{-A^2(t)} \;&=\; -\mathcal{R}^{\,b}STR_{<e>}\,[A(t),A'(t)e^{-A^2(t)}] \\ &+\;\mathcal{R}\int_{\mathbb{R}} \frac{tD_0}{\pi}e^{-(\,(\nabla^-_\partial)^2 + t^2D_0^2 + t^2\lambda^2\,)}d\lambda - \frac{d}{dt}\mathcal{R}^bTR_{<e>}[D, e^{-A^2(t)}_{1,1}P].\end{aligned}$$

*Proof.* Using Duhamel formula and Proposition 8 one gets:

$$\frac{d}{dt}\mathcal{R}^bSTR_{<e>}\,e^{-A^2(t)} = -\mathcal{R}^bSTR_{<e>}\int_0^1 e^{-uA^2(t)}(\,\frac{d}{dt}A^2(t)\,)e^{-(1-u)A^2(t)} =$$

$$-\mathcal{R}^bSTR_{<e>}\,[A(t),A'(t)e^{-A^2(t)}] -$$

$$\frac{i}{2\pi}\mathcal{R}STR_{<e>}\int_0^1\int_{\mathbb{R}}\big(\frac{\partial}{\partial\lambda}I(e^{-uA^2(t)};\lambda)I(\frac{d}{dt}A^2(t);\lambda)I(e^{-(1-u)A^2(t)};\lambda)\big)dud\lambda. \tag{44}$$

Due to the definition of $\nabla'$ we have:

$$I^2(\nabla'^{,+},\lambda) = (i\lambda + D_0)^{-1}(\nabla^-_\partial)^2(i\lambda + D_0).$$

From (42) and (43) one gets the two following formulas:

$$I(\frac{d}{dt}A^2(t);\lambda) = \begin{pmatrix} 2D_0^2t + 2t\lambda^2 & * \\ 0 & 2D_0^2t + 2t\lambda^2 \end{pmatrix}$$

$$I(e^{-uA^2(t)};\lambda) = \begin{pmatrix} (i\lambda + D_0)^{-1}e^{-u((\nabla^-_\partial)^2 + ut^2D_0^2 + ut^2\lambda^2)}\,(i\lambda + D_0) & * \\ 0 & e^{-u(\nabla^-_\partial)^2 - ut^2D_0^2 - ut^2\lambda^2} \end{pmatrix} \tag{45}$$

Then, a little computation (using the two previous equations) allows to see that:

$$-\frac{i}{2\pi}\mathcal{R}STR_{<e>}\int_0^1\int_{\mathbb{R}}\big(\frac{\partial}{\partial\lambda}I(e^{-uA^2(t)};\lambda)I(\frac{d}{dt}A^2(t);\lambda)I(e^{-(1-u)A^2(t)};\lambda)\big)dud\lambda =$$

$$\frac{1}{2\pi}\mathcal{R}TR_{<e>}\int_{\mathbb{R}}(2D_0^2+2t\lambda^2)(i\lambda+D_0)^{-1}e^{-(\nabla_{\partial}^{-})^2-t^2D_0^2-t^2\lambda^2}\,dud\lambda-$$
$$\frac{1}{2\pi}\mathcal{R}TR_{<e>}\int_{\mathbb{R}}(i\lambda+D_0)^{-1}\int_0^1 e^{-u((\nabla_{\partial}^{-})^2+t^2D_0^2+t^2\lambda^2)}(2D_0^2+2t\lambda^2)e^{-(1-u)((\nabla_{\partial}^{-})^2+t^2D_0^2+t^2\lambda^2)}dud\lambda.$$

Computing a little more one gets:

$$\frac{1}{2\pi}\mathcal{R}TR_{<e>}\int_{\mathbb{R}}(2D_0^2+2t\lambda^2)(i\lambda+D_0)^{-1}e^{-(\nabla_{\partial}^{-})^2-t^2D_0^2-t^2\lambda^2}\,dud\lambda=$$
$$\mathcal{R}TR_{<e>}\int_{\mathbb{R}}\frac{1}{\pi}tD_0e^{-(\nabla_{\partial}^{-})^2-t^2D_0^2-t^2\lambda^2}\,dud\lambda.$$

Then, using lemma 12 and Duhamel formula, one gets that:

$$\frac{1}{2\pi}\mathcal{R}TR_{<e>}\int_{\mathbb{R}}(i\lambda+D_0)^{-1}\int_0^1 e^{-u((\nabla_{\partial}^{-})^2+t^2D_0^2+t^2\lambda^2)}(2D_0^2+2t\lambda^2)e^{-(1-u)((\nabla_{\partial}^{-})^2+t^2D_0^2+t^2\lambda^2)}dud\lambda=$$
$$\frac{d}{dt}\mathcal{R}^bTR_{<e>}[D,e_{1,1}^{-A^2(t)}P].$$

Then, from the two previous equations and from (44) one obtains the Lemma. □

**Lemma 16.** *The following formula holds:*

$$\mathcal{R}^bSTR_{<e>}[A(t),A'(t)e^{-A^2(t)}]=d\mathcal{R}^bSTR_{<e>}\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}e^{-A^2(t)}$$
$$+\ \frac{1}{\pi}\int_{\mathbb{R}}\mathcal{R}TR_{<e>}\,tD_0\,e^{-(\nabla_{\partial}^{-})^2-t^2D_0^2-t^2\lambda^2}\,d\lambda.$$

*Proof.* Using the very definition of $A(t)$ one gets: $\mathcal{R}^bSTR_{<e>}[A(t),A'(t)e^{-A^2(t)}]=$

$$\mathcal{R}^bSTR_{<e>}[\begin{pmatrix}0 & tD^-\\ tD^+ & 0\end{pmatrix},\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}e^{-A^2(t)}]+$$
$$\mathcal{R}^bSTR_{<e>}[\nabla,A'(t)e^{-A^2(t)}]+\mathcal{R}^bSTR_{<e>}[\nabla'-\nabla,\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}e^{-A^2(t)}].$$

Now using the definition of $\nabla'$ and (45) (with $u=1$) one gets:

$$\begin{pmatrix}\frac{\partial}{\partial\lambda}I(\nabla'^{,+}-\nabla^+;\lambda) & 0\\ 0 & 0\end{pmatrix}I(A'(t)e^{-A^2(t)};\lambda)\equiv 0$$

therefore Proposition 8 shows that

$$\mathcal{R}^bSTR_{<e>}[\nabla'-\nabla,\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}e^{-A^2(t)}]\equiv 0.$$

Now using Proposition 8 and (45) (with $u=1$), a simple computation shows that:

$$\mathcal{R}^bSTR_{<e>}[\begin{pmatrix}0 & tD^-\\ tD^+ & 0\end{pmatrix},\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}e^{-A^2(t)}]=$$
$$\frac{i}{2\pi}\int_{\mathbb{R}}\mathcal{R}STR_{<e>}\begin{pmatrix}-itD_0+t\lambda & 0\\ 0 & itD_0+t\lambda\end{pmatrix}I(e^{-A^2(t)};\lambda)\,d\lambda\ =$$
$$\frac{1}{\pi}\int_{\mathbb{R}}\mathcal{R}TR_{<e>}\,tD_0\,e^{-(\nabla_{\partial}^{-})^2-t^2D_0^2-t^2\lambda^2}\,d\lambda.$$

Lastly, Lemma 14 shows that:

$$\mathcal{R}^b STR_{<e>}\,[\nabla, A'(t)e^{-A^2(t)}] = d\mathcal{R}^b STR_{<e>} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} e^{-A^2(t)}.$$

The Lemma is proved. □

By combining the definition of $\mathrm{Ch}_{<e>}\, A(t)$ and the Lemmas 15 and 16 one gets

$$\frac{d}{dt}\,\mathrm{Ch}_{<e>}\, A(t) = \frac{d}{dt}\big(\,\mathcal{R}^b TR_{<e>}\,[D^+, e_{1,1}^{-A^2(t)} P] + \mathcal{R}^b STR_{<e>}\, e^{-A^2(t)}\,\big) = \tag{46}$$

$$-d\mathcal{R}^b STR_{<e>} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} e^{-A^2(t)}.$$

As observed in section 5.2 of [9], $\mathcal{R}^b STR_{<e>} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} e^{-A^2(t)}$ is not integrable at $t = 0$ but we are going to show that we can add an exact form to it so that the sum does become integrable at $t = 0$. To this aim we shall need the following Lemma:

**Lemma 17.** *For any reals $s > 0$ and $v > 0$ one has:* $[\nabla^-; e^{-s(v(\nabla^-)^2 + D^+D^-)}\,] =$

$$-\int_0^s e^{-(s-u)(v(\nabla^-)^2 + D^+D^-)}\,[\nabla^-, v(\nabla^-)^2 + D^+D^-]e^{-u(v(\nabla^-)^2 + D^+D^-)}\,du.$$

*Proof.* Let us fix $v > 0$. Then one checks easily that $X(s) =$

$$[\nabla^-; e^{-s(v(\nabla^-)^2 + D^+D^-)}] + \int_0^s e^{-(s-u)(v(\nabla^-)^2 + D^+D^-)}\,[\nabla^-, v(\nabla^-)^2 + D^+D^-]e^{-u(v(\nabla^-)^2 + D^+D^-)}du$$

satisfies $(\frac{\partial}{\partial s} + v(\nabla^-)^2 + D^+D^-)X(s) \equiv 0$ and that $\lim_{s\to 0^+} X(s) = 0$. By a uniqueness argument one then gets the Lemma.

□

Now, following section 5.2 (and especially formulas (5.31) and (5.32)) of [9] one sees that modulo an integrable function at $t = 0$, $\mathcal{R}^b STR_{<e>} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} e^{-A^2(t)}$ is equal to

$$\frac{1}{t}\mathcal{R}^b TR_{<e>} \int_0^1 e^{-u((\nabla^-)^2 + t^2 D^+D^-)}[\nabla^-, (\nabla^-)^2 + t^2 D^+D^-]e^{-(1-u)((\nabla^-)^2 + t^2 D^+D^-)}du.$$

But applying Lemma 17 with $s = t^2$ and $v = \frac{1}{t^2}$ and using a linear change of variables to replace $\int_0^{t^2}$ by $\int_0^1$ one sees that the previous term is equal to

$$-\frac{1}{t}\mathcal{R}^b TR_{<e>}[\nabla^-, e^{-((\nabla^-)^2 + t^2 D^+D^-)}\,].$$

Hence, using Lemma 14 one deduces that modulo graded commutators

$$\mathcal{R}^b STR_{<e>} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} e^{-A^2(t)} + \frac{1}{t} d\,\mathcal{R}^b TR_{<e>} e^{-((\nabla^-)^2 + t^2 D^+D^-)}$$

is integrable at $t = 0$. Therefore, having in mind (46), one sees that we may write (up to graded commutators) that for any $t \in ]0,1]$, $\frac{d}{dt}\,\mathrm{Ch}_{<e>}\, A(t) = dF(t)$ where $t \to F(t)$ belongs $C^1([0,1]; \overline{\widehat{\Omega}}_*(T, \mathcal{B}_\Gamma^\infty))$. Since $t \to \mathrm{Ch}_{<e>}\, A(t)$ is continuous at $t = 0$ one obtains Step 4.

9.3. **Proof of Step 2.**

The proof of the Step 2 consists in the analysis of a transgression formula for the homotopy $u\mathbb{A}_t + (1-u)A(t)$ $(0 \leq u \leq 1)$ as $t \to +\infty$. One takes care of the details by using similar techniques as the ones used in the proof of Step 4 given above and also in section 14 of [17].

## 10. **General étale groupoids**

We follow Section 6 of [9]. Let $G$ be a smooth Hausdorff etale groupoid with units $G^{(0)}$, recall that the range maps $r : G \to G^{(0)}$ and the source maps $s : G \to G^{(0)}$ are (by assumption) local diffeomorphisms. We shall assume that $G^{(0)}$ is a smooth compact manifold possibly with boundary and that $G^{(0)}$ is endowed with a riemannian metric $g$ . Notice that the pull back by the map $s : G \to G^{(0)}$ of $g$ defines a riemannian metric on $G$, we shall denote by $\Delta$ the corresponding Laplace Beltrami operator on $G$.

We consider a fiber bundle $\sigma : L \to G^{(0)}$ such that each fiber $L_\theta = \sigma^{-1}(\theta)$ is a complete length space with metric $d_\theta$. We assume that $G$ acts isometrically, properly and cocompactly on $L$. Let $i : G \to L$ be a $G-$equivariant map ($i$ is not necessarily continuous) sending $G_\theta$ into $L_\theta$ for any $\theta \in G^{(0)}$. We assume that the preimage by $i$ of any compact set of $L$ has compact closure in $G$ and for any compact subset $K$ of $G^{(0)}$ $(\subset G)$, $i(K)$ has compact closure. One defines a length function on $G$ by

$$\forall \gamma \in G,\ l(\gamma) = d_{s(\gamma)}(i(s(\gamma)), i(\gamma))$$

where we recall that $\gamma$ and $s(\gamma)$ belong to $G_{s(\gamma)}$. Moreover we assume that

$$\exists C, N \in \mathbb{R}^+\ \forall \theta \in G^{(0)},\ \#\{\gamma \in G_\theta / \, l(\gamma) \leq R\} \leq C(1+R)^N. \tag{47}$$

This assumption is the generalization of the hypothesis "$\Gamma$ virtually nilpotent" in the case $G = T \rtimes \Gamma$, we recall that Section 6.1 of [9] explains to which extent these assumptions generalize to $G$ the ones made for $T \rtimes \Gamma$. Now we are going to define a smooth sub-algebra $\mathcal{T}^\infty(G)$ of $C_r^*(G)$ which is the analogue for $G$ of $\mathcal{T}^\infty$ for $T \rtimes \Gamma$. We fix a Haar system in $G$.

**Definition 14.** *We denote by $\mathcal{T}^\infty(G)$ the sub-algebra of $C_r^*(G)$ whose elements are the functions $f \in C^\infty(G)$ such that for any $p, q \in \mathbb{N}$ :*

$$\sup_{\gamma \in G} \big( (1 + l(\gamma) + l(\gamma^{-1}))^q \, |\Delta^p f(\gamma)| \big) < +\infty.$$

**Remark.** The fact that $\mathcal{T}^\infty(G)$ is an algebra is a consequence of the estimates: $l(\gamma \cdot \gamma') \leq l(\gamma) + l(\gamma')$.

Now for each $n \in \mathbb{N}$, we denote (following [9]) by $G^{(n)}$ the set of $n-$chains of composable elements of $G$:

$$G^{(n)} = \{(\gamma_1, \cdots, \gamma_n) \in G^n : \, s(\gamma_1) = r(\gamma_2), \cdots, s(\gamma_{n-1}) = r(\gamma_n) \, \}.$$

Then, for $(m, n) \in \mathbb{N}^2$, we denote by $\widetilde{\Omega}^{m,n}(G)$ the subset of sooth differential forms $\omega \in \Omega^m(G^{(n+1)})$ such that for any $p, q \in \mathbb{N}$:

$$\forall p, q \in \mathbb{N}, \quad \sup_{(\gamma_0, \cdots, \gamma_n) \in G^{(n+1)}} \big( (1 + l(\gamma_0) + \cdots + l(\gamma_n))^q \, |\Delta^p \omega(\gamma_0, \cdots, \gamma_n)| \big).$$

Then we denote by $\Omega^*(G)$ the quotient of $\Pi_{m,n \in \mathbb{N}} \, \Omega^m(G^{(n+1)})$ by the set of differential forms which are supported on $\{(\gamma_0, \cdots, \gamma_n) \in G^{(n+1)}, \ \gamma_j \text{ is a unit for some } j \, \}$.

Now we consider a smooth $G-$manifold with boundary $P$ (see section II.10.$\alpha$ of [7]). That is, first of all, there is a fiber bundle $\pi : P \to G^{(0)}$ whose fibers are manifolds with boundary which are transverse to $\partial P$ and of dimension $2k$. Given $\theta \in G^{(0)}$, we write $Z_\theta = \pi^{-1}(\theta)$. Putting

$$P \times_{G^{(0)}} G = \{(p,\gamma) \in P \times G : \; p \in Z_{r(\gamma)}\},$$

we have a map $P \times_{G^{(0)}} G \to P$, denoted $(p,\gamma) \to p\gamma$, such that $p\gamma \in Z_{s(\gamma)}$ and $(p\gamma_1)\gamma_2 = p(\gamma_1\gamma_2)$ for any $(\gamma_1,\gamma_2) \in G^{(2)}$. We assume that $P$ is a proper $G-$manifold, ie that the map $P \times_{G^{(0)}} G \to P \times P$ given by $(p,\gamma) \to (p, p\gamma)$ is proper. Then the groupoid $\mathcal{G} = P \rtimes G$ with underlying space $P \times_{G^{(0)}} G$, with space of units $\mathcal{G}^{(0)} = P$ and maps $r(p,\gamma) = p$ and $s(p,\gamma) = p\gamma$, is a proper groupoid. We also assume that $G$ acts cocompactly and freely on $P$, equivalently $\mathcal{G}$ is a free and cocompact groupoid. Let us fix $h \in C_c^\infty(\mathcal{G}^{(0)})$ (constant in the normal direction near the boundary) such that for all $\theta \in \mathcal{G}^{(0)}$

$$\sum_{\gamma \in \mathcal{G}^\theta} h(s(\gamma)) = 1.$$

Then we recall that Gorokhovsky and Lott defined a connection

$$\nabla^{can} : \; C_c^\infty(\mathcal{G}^{(0)}) \to \Omega_c^1(\mathcal{G}) \otimes_{C_c^\infty(\mathcal{G})} C_c^\infty(\mathcal{G}^{(0)})$$

of the form $\nabla^{can} = \nabla^{1,0} \oplus \nabla^{0,1}$ where for any $F \in C_c^\infty(\mathcal{G}^{(0)})$, $\nabla^{1,0}(F) \in \Omega^1(\mathcal{G}^{(0)})$ is the de Rham differential of $F$ and $\nabla^{0,1}(F) \in \Omega_c^{0,1}(\mathcal{G}^{(0)}) \otimes_{C_c^\infty(\mathcal{G})} C_c^\infty(\mathcal{G}^{(0)})$ is given by

$$\forall \gamma_0 \notin \mathcal{G}^{(0)}, \;\; (\nabla^{0,1}(F)\,)\,(\gamma_0) \;=\; F(r((\gamma_0))\, h(s(\gamma_0)).$$

Now we assume that the $2k-$dimensional fibers $\pi^{-1}(\theta)$, $\theta \in G^{(0)}$ carry a $G-$invariant spin structure and that there exists a $G-$invariant smooth family of $b-$metrics on the fibers $\pi^{-1}(\theta)$, $\theta \in G^{(0)}$. We denote the typical fiber of $\pi$ by $Z$ and by $S^Z = (S^Z)^+ \oplus (S^Z)^- \to P$ the associated ($\mathbb{Z}_2-$graded) spinor bundle. We consider also a $G-$equivariant complex hermitian vector bundle $\widehat{V} \to P$ endowed with a $G-$invariant $b-$hermitian connection $\widehat{\nabla}$ satisfying $\widehat{\nabla}_{x\partial_x} = 0$ on the boundary $\partial P$. We then set $\widehat{E} = S^Z \otimes \widehat{V} = \widehat{E}^+ \oplus \widehat{E}^-$ which defines a smooth family of $\mathbb{Z}_2-$graded hermitian Clifford modules in the fibers $\pi^{-1}(\theta)$, $\theta \in G^{(0)}$. Then we get a smooth family of $G-$invariant $\mathbb{Z}_2-$graded Dirac type operators

$$D(\theta) = \begin{pmatrix} 0 & D^-(\theta) \\ D^+(\theta) & 0 \end{pmatrix}, \; \theta \in G^{(0)}$$

acting fiberwise on $C_c^\infty(P, \widehat{E})$. Moreover in a collar neighborhood ($\sim [0,1] \times \partial\pi^{-1}(\theta) = \{(x,y)\}$) of $\partial\pi^{-1}(\theta)$ we may write:

$$D^+(\theta) = \sigma(x\partial_x + D_0(\theta)\,)$$

where $D_0(\theta)$ is the induced boundary Dirac type operator acting on

$$C^\infty(\partial\pi^{-1}(\theta), \; \widehat{E}^+_{|_{\partial\pi^{-1}(\theta)}}\,).$$

In the rest of this paper we shall make the following
**Hypothesis A** There exists a real $\epsilon > 0$ such that for any $\theta \in G^{(0)}$, the $L^2-$spectrum of $D_0(\theta)$ acting on $L^2(\partial\pi^{-1}(\theta), \; \widehat{E}^+_{|_{\partial\pi^{-1}(\theta)}}\,)$ does not meet $]-\epsilon, \epsilon[$.

We set

$C^\infty_{\mathcal{T}^\infty(G)}(P,\widehat{E}) = \mathcal{T}^\infty(G)\otimes_{C_c^\infty(G)} C_c^\infty(P,\widehat{E}),\ C^\infty_{\mathcal{T}^\infty(G)}(\partial P,\widehat{E}) = \mathcal{T}^\infty(G)\otimes_{C_c^\infty(G)} C_c^\infty(\partial P,\widehat{E}_{|\partial P}).$

As in Section 2, the $(D(\theta))_{\theta\in G^{(0)}}$ [resp. $(D_0(\theta))_{\theta\in G^{(0)}}$ ] induce a a left $\mathcal{T}^\infty(G)-$linear endomorphism of $C^\infty_{\mathcal{T}^\infty(G)}(P,\widehat{E})$ [resp. $C^\infty_{\mathcal{T}^\infty(G)}(\partial P,\widehat{E})$].

Now we fix a $G-$invariant horizontal distribution $T^HP$ such that

$$^bTP = T^HP \oplus {}^bT(P/T)$$

and, as in [9] and [27] (section 9) we consider the Bismut superconnection

$$\mathbb{A}_s^{Bismut} = sD + \nabla^u - \frac{1}{4s}c(\tau),\ s\in\mathbb{R}^{+*} \tag{48}$$

where $D$ is the $\mathcal{T}^\infty(G)-$linear Dirac operator introduced above, $c(\tau)$ denotes the Clifford multiplication by the curvature 2-form $\tau$ of $T^HP$ and $\nabla^u$ is a certain unitary connection. Then, as in [9] we consider for each real $s>0$ the superconnection

$$\mathbb{A}_s = \mathbb{A}_s^{Bismut} + \nabla^{0,1}$$

which sends $C^\infty_{\mathcal{T}^\infty(G)}(P,\widehat{E})$ into $\Omega^*(G)\otimes_{\mathcal{T}^\infty(G)} C^\infty_{\mathcal{T}^\infty(G)}(P,\widehat{E})$. By developing in a straightforward way a $b-$heat calculus as in [17] (section 10) one sees easily that $e^{-s^2D^2}\in\Psi^{-\infty}_{b,\mathcal{T}^\infty(G)}(P\,;\,\widehat{E})$ for any $s>0$ (the definition of this space is completely analogous to the one given in Section 3). Moreover, using a Duhamel expansion around $e^{-s^2D^2}$ one checks that for any $s>0$

$$e^{-\mathbb{A}_s^2}\in\Psi^{-\infty}_{b,\Omega^*(G)}(P\,;\,\widehat{E}).$$

Now as in [27] (section 10) we consider for any $s>0$ the induced boundary connection

$$\mathbb{B}_s = s\sigma D_0 + \nabla^{0,1}$$

where $D_0$ is the boundary Dirac operator of $D$ introduced above, $\nabla^u_\partial$ is a certain unitary connection and $c(\partial\tau)$ is the boundary connection of $c(\tau)$. We fix a function $\phi\in C_c^\infty(P)$ which is constant in the normal direction near the boundary such that

$$\forall p\in P,\quad \sum_{\gamma\in G^{\pi(p)}}\phi(p\gamma)=1.$$

Such a function $\phi$ was used in Sections 16 and 5 in the definition of the higher super traces. Then proceeding as in the proof of Proposition 6 we can show that the higher eta invariant

$$\widetilde{\eta}_{<e>} = \frac{2}{\sqrt{\pi}}\mathcal{R}\int_0^{+\infty} STR_{Cl(1)}(\,\frac{d\mathbb{B}_s}{ds}e^{-\mathbb{B}_s^2}\,)\,ds\in\Omega^*_{ab}(G)$$

is well defined where $\Omega^*_{ab}(G)$ denotes the quotient of $\Omega^*(G)$ by the closure of the space generated by the graded commutators. The differential of $\Omega^*(G)$ allows to endow $\Omega^*_{ab}(G)$ with the structure of a complex whose associated homology will denoted by $\widehat{H}_*(\Omega^*_{ab}(G))$.

Now we may state the higher local index theorem whose proof proceeds along the line of the proofs of Theorem 2 of [9], Theorem 13.6 of [17] and of Theorem 2

**Theorem 5.**

$$\lim_{s\to 0^+} \mathcal{R}^b STR_{<e>}\, e^{-\mathbb{A}_s^2} = \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) \in \Omega^*_{ab}(G)$$

*where $Z$ denotes the typical fiber of $\pi : P \to T$ and the connection $\nabla^{can}$ is defined in the section 6 of* [9].

**Corollary 3.** *Let $\Phi$ be a closed graded trace on $\Omega^*_{ab}(G)$ concentrated in the trivial conjugacy class. Then there is a current $\omega_\Phi$ on $M$ such that the following formula holds:*

$$< \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can})\,;\, \Phi >=< \widehat{A}(T\mathcal{F})\,\mathrm{ch}(\nabla^V)\,\omega_\Phi >$$

Recall we assume that the quotient $M = P/G$ is a smooth compact manifold with boundary, it inherits a foliation $\mathcal{F}$ whose leaves are the image by the map $P \to P/G$ of the fibers of the fibration $\pi : P \to G^{(0)}$. If we assume that $G^{(0)}$ admits a $G-$invariant riemannian metric $g$ then we may state for the $\mathcal{T}^\infty(G)-$linear operator $D^+$ a result completely analogous to Theorem 9 so that the index class $\mathrm{Ind}\, D^+$ is well defined in $K_0(\mathcal{T}^\infty(G))$. The higher index $\mathrm{Ind}_\Phi(D^+)$ associated to a closed graded trace on $\Omega^*(G)$ is defined as in Definition 10. In this general case, proceeding as in Section 7, we obtain the following analogue of Theorem 3

**Theorem 6.** *Assume that $G^{(0)}$ admits a $G-$invariant riemannian metric $g$, then:*

- *By universality there is a natural morphism of algebras*
$$j : \Omega^*(\mathcal{T}^\infty(G)) \to \Omega^*_{ab}(G).$$
- *The following formula holds in the homology groups $\widehat{H}_*(\Omega^*_{ab}(G))$ :*
$$j_{<e>}(\mathrm{Ch\ Ind}\, D^+) = \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) - \frac{1}{2}\widetilde{\eta}_{<e>} \in \widehat{H}_*(\Omega^*_{ab}(G)).$$
- *Let $\Phi$ denote a closed graded trace on $\Omega^*(G)$. Then*
$$\mathrm{Ind}_\Phi(D^+) =< \widehat{A}\,(T\mathcal{F})\,\mathrm{ch}(\nabla^V)\,,\, \omega_\Phi > - < \frac{1}{2}\widetilde{\eta}_{<e>}\,;\, \Phi > .$$

In the general case (i.e. when $G^{(0)}$ does not admit any a $G-$invariant riemannian metric) we define $\mathrm{Ch}_{<e>}\ \mathrm{Ind}\, D^+) \in \Omega^*_{ab}(G)$ exactly as in Definition 13.

Then, we have the following Theorem whose proof is completely analogous to the one of Theorem 4.

**Theorem 7.**
*1) The following formula holds in the homology $\widehat{H}_*(\Omega^*_{ab}(G))$ (see Section 5.1):*

$$\mathrm{Ch}_{<e>}\,(\mathrm{Ind}\, D^+) = \int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widehat{V}})\,\mathrm{ch}(\nabla^{can}) - \frac{1}{2}\widetilde{\eta}_{<e>} \in \widehat{H}_*(\Omega^*_{ab}(G))$$

*2). Let $\Phi$ a closed graded trace on $\Omega^*_{ab}(G)$, concentrated on the trivial conjugacy class. Let $\widetilde{\Phi}$ be its extension to $\Psi^{-\infty,\delta}_{\Omega^*_{ab}(G)}$. Then one has:*

$$\mathrm{Ind}_\Phi(D^+) :=< \mathrm{Ch}(\mathrm{Ind}\, D^+)\,;\, \widetilde{\Phi} >=< \widehat{A}(T\mathcal{F})\,\mathrm{ch}(\nabla^V)\,,\, \omega_\Phi > - < \frac{1}{2}\widetilde{\eta}_{<e>}\,;\, \Phi >$$

## 11. **Applications to foliations**

We give an application of the previous theorem to a class of foliations. Let $(Y,\mathcal{F})$ be a foliated manifold with boundary where the leaves of $\mathcal{F}$ are transversal to the boundary $\partial Y$ and of dimension $2k$. We assume that the holonomy groupoid of $(Y,\mathcal{F})$ is Hausdorff, that ${}^bT\mathcal{F}$ carries a riemannian $b-$metric and that $\mathcal{F}$ admits a leafwise spin structure, we shall denote by $S^{\mathcal{F}}$ the spinor bundle. Moreover, we consider $V \to Y$ a complex hermitian vector bundle endowed with a hermitian connection. These data fix a $b-$Dirac type leafwise operator $D$ acting on the sections of $V\otimes S^{\mathcal{F}}$. Let $H$ denote the holonomy groupoid of $(Y,\mathcal{F})$ and let $s : H \to Y$ denote the source map. By lifting $D$ to $H$ we get a $H-$invariant Dirac type operator on $H$ still denoted $D$. We shall denote by $D_0$ the boundary operator on $\partial H$ of $D$ and assume that there exists $\epsilon > 0$ such that for any $y \in \partial Y$, the $L^2-$spectrum of $D_0$ acting on $L^2(\partial s^{-1}(y)\,,\,(V\otimes S^{\mathcal{F}})_{|\partial s^{-1}(y)}\,)$ does not meet $]-\epsilon,\epsilon[$. Then one may associate to $D$ an index class $\operatorname{Ind} D^+ \in K_0(C^*_r(H))$, following an argument of [7], we are going to use a complete transversal $T$ and the associate etale groupoid $H^T_T = G$ so as to interpret (via a Morita equivalence invariance argument) $\operatorname{Ind} D^+$ as an element of $K_0(C^*_r(G))$ and to apply to it Theorem 6. In fact we need to make the following assumptions:

*1)* There exists a smooth compact submanifold (possibly with boundary) $T$ of $Y$ which defines a complete transversal of $(Y,\mathcal{F})$ and we denote by $G = H^T_T$ the etale groupoid formed of the elements of $H$ having their extremities in $T$.

*2)* Set $P = H_T = s^{-1}(T)$. We assume that the restriction denoted $\pi = s_{|P} : P \to T$ is a fibration over $T$.

*3)* Denote by $i$ the inclusion $G \to H_T = s^{-1}(T)$ and set $L = P = s^{-1}(T)$ and $\sigma = s_{|P}$. Recall that for each $\gamma \in G$, $s^{-1}(s(\gamma)) = G_{s(\gamma)}$ is a complete length metric space. Then we define a length function $l$ on $G$ by setting for any $\gamma \in G$, $l(\gamma) = d_{s(\gamma)}(\,i(s(\gamma)), i(\gamma)\,)$. We then assume that (47) holds.

With these assumptions we consider the algebra $\mathcal{T}^\infty(G)$ and the space $\Omega^*(G)$ as defined previously.

Let $\int_{\mathcal{F}}$ denote the Haefliger integration map from $\Omega^*(Y)$ to $\Omega^*(T)/V$ where $V$ denote the sub-vector space of $\Omega^*(T)$ generated by the differential forms $\omega - h(\omega)$, $h \in G$. Then Theorem 6 implies the following

**Theorem 8.** *Let $\tau$ be a holonomy-invariant closed transverse current of $(Y,\mathcal{F})$; it induces a closed graded trace on $\Omega^*(G)$ still denoted $\tau$. Denote by $\widetilde{\tau}$ its extension to $\Psi^{-\infty,\delta}_{\Omega^*_{ab}(G)}$. We then have:*

$$< \operatorname{Ch}\, \operatorname{Ind} D^+)\,;\,\widetilde{\tau} >=< \int_{\mathcal{F}} \widehat{A}\,(T\mathcal{F})\,\operatorname{ch}(\nabla^V)\,;\,\tau > - < \frac{1}{2}\widetilde{\eta}_{<e>}\,;\,\tau > .$$

## 12. **Appendix A: the rapidly decreasing $b$-calculus**

In order to define the rapidly decreasing $b$-calculus we introduce as before an auxiliary metric $\hat{g}$ on $\widehat{M}$ for which $\widehat{M}$ and $\Gamma$ ($\Gamma$ viewed as a metric space with respect to the word-metric) become *quasi-isometric*. The metric $\hat{g}$ is simply the lift to $\widehat{M}$ of an ordinary metric on $M$. In the sequel we denote by $d(\cdot,\cdot)$ the distance function associated to $\hat{g}$.

Let $\beta_\pi : [\widehat{M}\times_\pi \widehat{M}, B] \to \widehat{M}\times_\pi \widehat{M}$ be the blow-down map associated to the fiber-$b$-stretched product.

We consider the fibration $\pi : \widehat{M} \times_\pi \widehat{M} \to T$. Let $(U_j)_{1\leq j\leq l}$ be a finite open cover of $T$ such that for each $j \in \{1,\dots,l\}$, the fibration $\pi : \widehat{M} \times_\pi \widehat{M} \to T$ is trivial over $U_j$. For each $j \in \{1,\dots,l\}$ let us choose a trivialization $\chi_j : \pi^{-1}(U_j) \sim Z \times Z \times U_j$ where $Z$ denotes the typical fiber of $\pi : \widehat{M} \to T$. Let $X_j$ be any smooth vector field on $U_j \subset T$ with compact support on $U_j$, we still denote by $X_j$ the induced vector field on $Z \times Z \times U_j$, then $\chi_j^*(X_j)$ defines a differential operator acting on $C^\infty(\widehat{M} \times_\pi \widehat{M}\,, \widehat{E} \boxtimes \widehat{E}^\star$. Moreover, for any $p, q \in \mathbb{N}$ the family of fiberwise operators $D^p(\theta)$ (resp. $D^q(\theta)$) defines a differential operator $D_l^p(\cdot)$ (resp. $D_r^q(\cdot)$) acting on any Schwartz kernel $K(z,w) \in C^\infty(\widehat{M} \times_\pi \widehat{M}\,, \widehat{E} \boxtimes \widehat{E}^\star)$ by the formula: $D_l^p(\cdot) \cdot K(z,w) = D^p(\pi(z)) \circ K(z,w)$ (resp. $D_r^p(\cdot) \cdot K(z,w) = K(z,w) \circ D^q(\pi(w))$).

**Definition 15.**
*1. We shall denote by* ${}^b\mathrm{Op}\,(\widehat{M} \times_\pi \widehat{M})$ *the algebra of differential operators acting on* $C^\infty(\widehat{M} \times_\pi \widehat{M}\,, \widehat{E} \boxtimes \widehat{E}^\star)$ *which is generated by all the above operators* $\chi_j^*(X_j)$, $D^p(\cdot)$, $D^q(\cdot)$.
*2. We shall denote by* ${}^b\mathrm{Op}\,([\widehat{M} \times_\pi \widehat{M}, B])$ *the span over* $C^\infty([\widehat{M} \times_\pi \widehat{M}, B])$ *of* $\beta_\pi^*({}^b\mathrm{Op}\,(\widehat{M} \times_\pi \widehat{M})$ *where* $\beta_\pi^*$ *denotes the lift of the blow-down map* $\beta_\pi$.

**Definition 16.** *Let* $\Gamma$ *be virtually nilpotent and let* $P \in \Psi^m_{b,\rtimes}(\widehat{M}, \widehat{E})$. *We shall say that* $P$ *is rapidly decreasing outside an* $\epsilon$*-neighborhood of the lifted fiber-diagonal in* $\widehat{M} \times_\pi \widehat{M}$ *if for each* $Q \in^b \mathrm{Op}\,([\widehat{M} \times_\pi \widehat{M}, B])$ *and any* $q \in \mathbb{N}$ *we can find a constant* $C_{Q,q} > 0$ *such that* $\forall p \in [\widehat{M} \times_\pi \widehat{M}, B]$ *such that* $d(z,z') > \epsilon$ *where* $(z,z') = \beta_\pi(p)$

$$|Q(K_P)(p)|(1 + d(z,z'))^q < C_{Q,q}.$$

**Definition 17.** *The elements in* $\Psi^*_{b,\rtimes}(\widehat{M}, \widehat{E})$ *which are rapidly decreasing outside the lifted diagonal form a subalgebra which is, by definition, the rapidly decreasing algebra* $\Psi^*_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$.

**Definition 18.** *We denote by* $H^\infty_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$ *the subset of the elements* $s \in H^\infty_{b,\mathrm{loc}}(\widehat{M}, \widehat{E})$ *such that for any* $\phi \in C_c^\infty(\widehat{M})$ *and any integers* $p, q \in \mathbb{N}$:

$$\sup_{\gamma \in \Gamma} \big( (1 + ||\gamma||)^p ||\phi\,(\gamma \cdot s)||_{H^q_b(\widehat{M}, \widehat{E})} \big) < +\infty$$

*where* $H^q_{b,\Gamma}(\widehat{M}, \widehat{E})$ *is defined using* $\Gamma-$*equivariant* $b-$*differential operators of order* $\leq q$ *acting on* $C^\infty(\widehat{M}, \widehat{E})$ *(nevertheless, notice that the sections* $\phi\,(\gamma \cdot s)$ *have compact support in supp*$\phi$.

We shall now enlarge the algebra $\Psi^*_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$ and define the rapidly decreasing $b-$calculus with bounds

$$\Psi^{*,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E}) := \Psi^*_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E}) + \Psi^{-\infty,\epsilon}_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E}) + \Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E})\,.$$

**Definition 19.**
*1. We denote by* $\Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$ *the set of left* $\mathcal{T}^\infty-$*linear operators acting on* $H^\infty_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E})$ *defined by:*

$$s \to \int_{\pi^{-1}(\theta)} K(z,y) s(y) dVol^b_{\pi^{-1}(\theta)} \;=\; K(s)(z)$$

*where the Schwartz kernel* $K(z,y) \in \rho_{lb}^{\epsilon}\rho_{rb}^{\epsilon} H^{\infty}_{b,\mathrm{loc}}(\widehat{M} \times_{\pi} \widehat{M}, \widehat{E} \boxtimes \widehat{E}^*)$ *is* $\Gamma-$*invariant and such that for any operator* $P \in {}^{b}\mathrm{Op}\,(\widehat{M} \times_{\pi} \widehat{M})$ *and any* $p \in \mathbb{N}$*:*

$$\sup_{R>1} R^p ||1_{d(z,y)>R}\,(\,P(\rho_{lb}^{-\epsilon}\rho_{rb}^{-\epsilon}K(\cdot,\cdot)\,)||_{L^2_b(\widehat{M}\times_{\pi}\widehat{M}\,,\,\widehat{E}\boxtimes\widehat{E}^*)} < +\infty \tag{49}$$

*2. Proceeding as in Remark 8, one can define the space of operators* $\Psi^{-\infty,\epsilon}_{b,\mathcal{T}^{\infty}}(\widehat{M},\widehat{E})$ *by considering the doubled space* $\mathcal{D}_{bf}([\widehat{M} \times \widehat{M}, B])$ *and the* $\Gamma$*-invariant elements of*

$$H^{\infty}_{b,loc}(\mathcal{D}_{bf}([\widehat{M} \times \widehat{M}, B]);(\widehat{E} \boxtimes \widehat{E}^{\star})_{\mathcal{D}})$$

*satisfying estimates analogous to (49).*

## 13. Appendix B: $b$-smoothing operators with differential form coefficients.

Let $\epsilon \in (0,1)$ and let

$$\mathcal{O}(\mathrm{bf}) = \{\, p \in [\widehat{M} \times_{\pi} \widehat{M}, B] \mid d(\beta_{\pi}(p), B) < \epsilon\,\}.$$

In $\mathcal{O}(\mathrm{bf})$ the variables $r = x + x'$ and $\tau = (x - x')/(x + x')$ together with the boundary variables $(y, y')$ (see [26] Ch. 4) can be used. We set $\mathcal{C}(\mathrm{bf}) := [\widehat{M} \times_{\pi} \widehat{M}, B] \setminus \mathcal{O}(\mathrm{bf})$ and we identify it with its image in $\widehat{M} \times_{\pi} \widehat{M}$ under the blow-down map $\beta_{\pi}$.

We shall now give a precise definion of the 3 spaces:

$$\Psi^{-\infty}_{b,\widehat{\Omega}_*(T,\mathcal{B}^{\infty}_{\Gamma})}(\widehat{M}\,;\,\widehat{E})\,, \quad \Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^{\infty}_{\Gamma})}(\widehat{M}\,;\,\widehat{E}) \quad \Psi^{-\infty,\delta}_{\widehat{\Omega}_*(T,\mathcal{B}^{\infty}_{\Gamma})}(\widehat{M}\,;\,\widehat{E})$$

**Definition 20.** *For any* $(k,l) \in \mathbb{N} \times \mathbb{N}$, $\Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^{\infty}_{\Gamma})}(\widehat{M}\,;\,\widehat{E})(k,l)$ *denotes the set of left* $\widehat{\Omega}_*(T,\mathcal{B}^{\infty}_{\Gamma})$*-linear operators:*

$$\widehat{\Omega}_*(T,\mathcal{B}^{\infty}_{\Gamma}) \otimes_{\mathcal{T}^{\infty}} H^{\infty}_{b,\mathcal{T}^{\infty}}(\widehat{M},\widehat{E}) \longrightarrow \widehat{\Omega}_*(T,\mathcal{B}^{\infty}_{\Gamma}) \otimes_{\mathcal{T}^{\infty}} H^{\infty}_{b,\mathcal{T}^{\infty}}(\widehat{M},\widehat{E})$$

*whose Schwartz kernel* $K$ *can be written in the form*

$$K(z,w) = \sum_{g_1,\cdots,g_l \in \Gamma} dg_1 \cdots dg_l\, K_{g_1,\cdots,g_l}(z,w)$$

*where for any* $(z,w) \in \widehat{M} \times \widehat{M}$, $K_{g_1,\cdots,g_l}(z,w) \in \bigwedge^k(T^*_{\pi(z)}T) \otimes Hom(\widehat{E}_w\,,\,\widehat{E}_z)$*, vanishes for* $\pi(z)(g_1\cdots g_l)^{-1} \neq \pi(w)$ *and is such that*

$$R^*_{((g_1\cdots g_l),e)} K_{g_1,\cdots,g_l} \;:=\; K_{g_1,\cdots,g_l}(\cdot g_1 \cdots g_l, \cdot)$$

*defines an element in* $\Psi^{-\infty,\delta}_{eb,\pi}(\widehat{M},\widehat{E})$*, the extended fiber-b-calculus with bounds. We require these kernels to satisfy the following estimates:*

*1] If* $\mathcal{C}(\mathrm{bf}) := [\widehat{M} \times_{\pi} \widehat{M}; B] \setminus \mathcal{O}(\mathrm{bf})$ *then for any fundamental domain* $A$ *of* $\widehat{M}$*, for any* $P \in^{b} \mathrm{Op}\,(\widehat{M} \times_{\pi} \widehat{M})$ *and any integer* $p > 1$

$$\sup_{\mathcal{C}(\mathrm{bf})} \big[d(zg_1\cdots g_l, A) + ||g_2|| + \cdots + ||g_l|| + d(w, Ag_1^{-1})\big]^p |P_{z,w} R^*_{((g_1,\cdots,g_l),e)} K_{g_1,\cdots,g_l}|$$

*is finite.*

*2] For any* $P \in \mathrm{Op}\,(\partial\widehat{M} \times_{\pi} \partial\widehat{M})$ *and any integers* $p, q_1, q_2 \in \mathbb{N}$*, we have*

$$\sup_{\mathcal{O}(\mathrm{bf})} \big[d(yg_1\cdots g_l, A) + ||g_2|| + \cdots + ||g_l|| + d(y', Ag_1^{-1})\big]^p |P\partial_{\tau}^{q_1}\partial_r^{q_2} R^*_{((g_1,\cdots,g_l),e)} K_{g_1,\cdots,g_l}|$$

*is finite.*
*3] We set:*

$$\Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{N}\,;\,\widehat{F}) = \oplus_{k,l\in\mathbb{N}}\Psi^{-\infty,\delta}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{N}\,;\,\widehat{F})(k,l).$$

*4] Proceeding as in Section 4.2, one defines in a similar way* $\Psi^{-\infty,\delta}_{\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{N}\,;\,\widehat{F})$.

5] The space $\Psi^{-\infty}_{b,\widehat{\Omega}_*(T,\mathcal{B}^\infty_\Gamma)}(\widehat{N}\,;\,\widehat{F})$ is defined as in 1], 2] above but with $C^\infty_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ appearing instead of $H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E})$ and with the kernels $R^*_{((g_1\cdots g_l),e)}K_{g_1,\dots,g_l}$ in $\Psi^{-\infty}_{eb,\pi}(\widehat{M}\widehat{E})$.

## 14. **Appendix C: a proof of theorem 3**

In this section we shall give a proof of Theorem 3. The proof is quite parallel to the proof of Theorem 14.1 of [17] so we shall only sketch the main lines of the proof. Notice that since [17] deals with *right*-modules there is in [17] a grading $\Upsilon$ in front of the Lott's connection, whereas here, since we deal with left $\mathcal{T}^\infty-$modules there is no such grading $\Upsilon$.

First of all we need the following decomposition theorem:

**Theorem 9.** *We can find $\epsilon > 0$, $\mathcal{L}_\infty$ [resp. $\mathcal{N}_\infty$] a sub-$\mathcal{T}^\infty-$module projective of finite rank of $x^\epsilon H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^+)$ [resp. $x^\epsilon H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^-)$] with the following properties:*
*1) $\mathcal{L}_\infty$ is free and $D^+(\mathcal{L}_\infty)\subset\mathcal{N}_\infty$.*
*2) As Frechet spaces:*

$$\mathcal{L}_\infty\oplus^\perp\mathcal{L}_\infty^\perp = x^\epsilon H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^+),\quad \mathcal{N}_\infty\oplus D^+(\mathcal{L}_\infty^\perp) = x^\epsilon H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^-).$$

*3) The orthogonal projection $\Pi_{\mathcal{L}_\infty}$ of $H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^+)$ onto $\mathcal{L}_\infty$ and the projection $\Pi_{\mathcal{N}_\infty}$ of $H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^-)$ onto $\mathcal{N}_\infty$ along $D^+(\mathcal{L}_\infty^\perp)$ are operators in $\Psi^{-\infty,\epsilon}_{\mathcal{T}^\infty}(\widehat{M},\widehat{E})$.*
*4) As Banach spaces*

$$(\,C^0(T)\rtimes_r\Gamma\otimes_{\mathcal{T}^\infty}\mathcal{L}_\infty\,)\oplus(\,\overline{C^0(T)\rtimes_r\Gamma\otimes_{\mathcal{T}^\infty}{\mathcal{L}_\infty}^\perp}\,) = L^2_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M},\widehat{E})$$

$$(\,C^0(T)\rtimes_r\Gamma\otimes_{\mathcal{T}^\infty}\mathcal{N}_\infty\,)\oplus(\,\overline{C^0(T)\rtimes_r\Gamma\otimes_{\mathcal{T}^\infty}D^+(\,{\mathcal{L}_\infty}^\perp\,)}\,) = H^{-1}_{b,C^0(T)\rtimes_r\Gamma}(\widehat{M},\widehat{E}).$$

*5) The operator*

$$D^+ : {\mathcal{L}_\infty}^\perp\to D^+({\mathcal{L}_\infty}^\perp)$$

*is invertible for the Frechet topologies; the induced operator*

$$D^+ : \overline{C^0(T)\rtimes_r\Gamma\otimes_{\mathcal{T}^\infty}{\mathcal{L}_\infty}^\perp}\to\overline{C^0(T)\rtimes_r\Gamma\otimes_{\mathcal{T}^\infty}D^+(\,{\mathcal{L}_\infty}^\perp\,)}$$

*is invertible*
*6) The operator $(D^+)^{-1}\circ(\mathrm{Id}-P_{\mathcal{N}_\infty})$ belongs to $\Psi^{-1,\epsilon}_{b,\mathcal{T}^\infty}$.*

As a consequence of the Theorem we immediately obtain:

$$\operatorname{Ind} D^+ = [\mathcal{L}_\infty]-[\mathcal{N}_\infty]\in K_0(\mathcal{T}^\infty)\simeq K_0(C^0(T)\rtimes_r\Gamma),$$

as already stated in subsection 7.2.

With the notations of Theorem 9, we set:

$$\mathcal{H}^+ = H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^+)\oplus\mathcal{N}_\infty = \mathcal{L}_\infty^\perp\oplus\mathcal{L}_\infty\oplus\mathcal{N}_\infty$$

$$\mathcal{H}^- = H^\infty_{b,\mathcal{T}^\infty}(\widehat{M},\widehat{E}^-)\oplus\mathcal{L}_\infty = D^+(\mathcal{L}_\infty^\perp)\oplus\mathcal{N}_\infty\oplus\mathcal{L}_\infty$$

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-.$$

Then we define, for each real $\alpha > 0$, the operator $\mathcal{R}^+_\alpha : \mathcal{H}^+ \to \mathcal{H}^-$ by:

$$\mathcal{R}^+_\alpha (f \oplus n) = (D^+ f + \alpha n) \oplus \alpha \Pi_{\mathcal{L}_\infty} f$$

where $f \in \mathcal{L}^\perp_\infty \oplus \mathcal{L}_\infty$ and $n \in \mathcal{N}_\infty$.

Next we define the operator $\mathcal{R}^-_\alpha : \mathcal{H}^- \to \mathcal{H}^+$ by:

$$\mathcal{R}^-_\alpha (g \oplus l) = (D^- g \oplus \alpha l) \oplus \alpha \Pi_{\mathcal{N}_\infty} g.$$

where $g \in H^\infty_{b,\mathcal{T}^\infty}(\widehat{M}, \widehat{E}^-)$ and $l \in \mathcal{L}_\infty$. Finally we define:

$$\mathcal{R}_\alpha = \begin{pmatrix} 0 & \mathcal{R}^-_\alpha \\ \mathcal{R}^+_\alpha & 0 \end{pmatrix}$$

Now we set $\mathcal{F}_\infty = \mathcal{L}_\infty \oplus \mathcal{N}_\infty$ and we define a $\mathcal{T}^\infty$−connection $\nabla_{\mathcal{F}_\infty}$ on $\mathcal{F}_\infty$ by compressing $\mathbb{A}_1 - D + \frac{1}{4}c(\tau)$ by $\Pi_{\mathcal{L}_\infty} \oplus \Pi_{\mathcal{N}_\infty} = \Pi_{\mathcal{F}_\infty}$. Then we observe that for each real $u > 0$

$$(\mathbb{A}_u - uD \oplus \nabla_{\mathcal{F}_\infty}) + u\mathcal{R}_\alpha$$

defines a superconnection operator on $\mathcal{H}$ and we set:

$${}^b\mathrm{ch}_{u,\alpha}(\widehat{E}_{\mathcal{T}^\infty}) = \mathcal{R}({}^bSTR_{<e>}\, e^{-((\mathbb{A}_u - D \oplus \nabla_{\mathcal{F}_\infty}) + u\mathcal{R}_\alpha)^2}).$$

Clearly we have:

$$(50) \qquad {}^b\mathrm{ch}_{u,0}(\widehat{E}_{\mathcal{T}^\infty}) = \mathcal{R}({}^bSTR_{<e>}\, e^{-\mathbb{A}_u^2}) - \mathcal{R}({}^bSTR_{<e>}\Pi_{\mathcal{F}_\infty} e^{-\nabla^2_{\mathcal{F}_\infty}} \Pi_{\mathcal{F}_\infty})$$

Now using Theorem 9 3] one sees that

$${}^bSTR\, \Pi_{\mathcal{F}_\infty} e^{-\nabla^2_{\mathcal{F}_\infty}} \Pi_{\mathcal{F}_\infty} = STR_{<e>}\, \Pi_{\mathcal{F}_\infty} e^{-\nabla^2_{\mathcal{F}_\infty}} \Pi_{\mathcal{F}_\infty}.$$

From Theorem 9 3] we get

$$j_{<e>}(\mathrm{Ch}\,\mathrm{Ind}\, D^+) \;=\; j_{<e>}(\mathrm{Ch}([\mathcal{L}_\infty]) - \mathrm{Ch}([\mathcal{N}_\infty])).$$

By using Proposition 11 3) and carrying out a tedious computation we prove easily the following lemma

**Lemma 18.**

$$STR^{alg}_{<e>} e^{-\nabla^2_{\mathcal{F}_\infty}} = STR_{<e>}\Pi_{\mathcal{F}_\infty} e^{-\nabla^2_{\mathcal{F}_\infty}} \Pi_{\mathcal{F}_\infty}.$$

The previous lemma and Proposition 11 thus imply that

$$j_{<e>}(\mathrm{Ch}([\mathcal{L}_\infty]) - \mathrm{Ch}([\mathcal{N}_\infty])) \;=\; \mathcal{R}\, STR_{<e>}\Pi_{\mathcal{F}_\infty} e^{-\nabla^2_{\mathcal{F}_\infty}} \Pi_{\mathcal{F}_\infty} \in \overline{H}_*(T, \mathcal{B}^\infty_\Gamma).$$

From the last three equations we deduce that

$$j_{<e>}(\mathrm{Ch}\,\mathrm{Ind}\, D^+) \;=\; \mathcal{R}\, STR_{<e>}\Pi_{\mathcal{F}_\infty} e^{-\nabla^2_{\mathcal{F}_\infty}} \Pi_{\mathcal{F}_\infty} \in \overline{H}_*(T, \mathcal{B}^\infty_\Gamma).$$

Hence from equations (33),(50) and the previous one we get $j_{<e>}(\mathrm{Ch}\,\mathrm{Ind}\, D^+) =$

(51)

$$\int_Z \phi\, \widehat{A}\,(\nabla^{TZ})\, \mathrm{ch}(\nabla^{\widehat{V}})\, \mathrm{ch}(\nabla^{can}) - \frac{1}{2}\int_0^u \widetilde{\eta}_{<e>}(s) ds - d\int_0^u \mathcal{R}({}^bSTR_{<e>}(\frac{d\mathbb{A}_s}{ds} e^{-\mathbb{A}_s^2}))ds - {}^b\mathrm{ch}_{u,0}(\widehat{E}_{\mathcal{T}^\infty})$$

We shall prove Theorem 3 by taking the limit $u \to +\infty$ of the right handside of the previous equation and showing that ${}^b\mathrm{ch}_{u,0}(\widehat{E}_{\mathcal{T}^\infty}) \to 0$ modulo $d\overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma)$ as $u \to +\infty$. Now, as in [17] pages 100 and 101 one can show that

$$ {}^b\mathrm{ch}_{u,\alpha}(\widehat{E}_{\mathcal{T}^\infty}) - {}^b\mathrm{ch}_{u,0}(\widehat{E}_{\mathcal{T}^\infty}) \in d\overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma) \tag{52} $$

If $\alpha$ is large enough then $\mathcal{R}^+_\alpha : \mathcal{H}^+ \to \mathcal{H}^-$ is invertible and its inverse belongs to the $b - \mathcal{T}^\infty-$calculus with bounds. Thus for $\alpha > 0$ large and $u > 1$ we consider the following connection on $\mathcal{H}^+$:

$$ \nabla_{u,\mathcal{H}^+} = (\mathcal{R}^+_\alpha)^{-1} \circ (\mathbb{A}_u - uD \oplus \nabla_{\mathcal{F}_\infty})_{|\mathcal{H}^+} \circ \mathcal{R}^+_\alpha. $$

Then we define a new connection $\nabla_{u,\mathcal{H}}$ on $\mathcal{H}$ by setting

$$ \nabla_{u,\mathcal{H}} = \nabla_{u,\mathcal{H}^+} \oplus -(\mathbb{A}_u - uD \oplus \nabla_{\mathcal{F}_\infty})_{|\mathcal{H}^-} $$

The two superconnections $\nabla_{u,\mathcal{H}}$ and $\mathbb{A}_u - uD \oplus \nabla_{\mathcal{F}_\infty}$ are of course homotopic through the path of connections parametrized by $\vartheta \in [0,1]$:

$$ \vartheta \nabla_{u,\mathcal{H}} + (1 - \vartheta)(\mathbb{A}_u - uD \oplus \nabla_{\mathcal{F}_\infty}). $$

Now, proceeding as in [17] pages 102-103, one first proves that, modulo $d\overline{\widehat{\Omega}}_*(T, \mathcal{B}^\infty_\Gamma)$:

$$ {}^b\mathrm{ch}_{u,\alpha}(\widehat{E}_{\mathcal{T}^\infty}) = {}^b\mathcal{R}STR\, e^{-(\nabla_{u,\mathcal{H}} + u\mathcal{R}_\alpha)^2} + B_2(u,\alpha) $$

Then, still as in [17] pages 102-103, one proves that

$$ \lim_{u \to +\infty} {}^b\mathcal{R}STR\, e^{-(\nabla_{u,\mathcal{H}} + u\mathcal{R}_\alpha)^2} = 0, \; \lim_{u \to +\infty} B_2(u,\alpha) = 0. $$

Theorem 3 is then a consequence of the equations (52), (51) and of the three previous ones.

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